A SunCam online continuing education course

Structural Nonlinearity: Analyzing Nonlinearity

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Introduction

Nonlinear analyses can be humbling. Engineering judgment is a vital part of the analysis and evaluation of structures, but it is often difficult to determine the validity and finality of nonlinear results. Nonlinearity isn't predictable so how can engineering judgment be applied, and how can nonlinear results be checked?

Nonlinear analysis can be like driving to an unknown destination. The driver knows how to operate the vehicle, knows to expect surprises and difficulties along the way, and knows that routes can be changed at any point; but how will the driver know how to adjust the route, or even if or when they have arrived at the destination?

This course aims to educate the reader on some of the nuances of nonlinear analyses and provides general recommendations for analyzing nonlinear structural systems. Several detailed examples are shown to illustrate the nature, tendencies, and perils of nonlinear analysis.

The scope of this course is limited to statics and generally presents concepts and examples as planar to leave the complexity to the topic at-hand.

> Structural movement includes deflections, slopes, displacements, and rotations of structures and their members. The term *distort* will be used as appropriate in this document to substitute for structural movement.
>
> The term *P-delta* is used in this document to collectively refer to both P-δ and P-Δ effects of geometric nonlinearity.
>
> The material nonlinearity discussed in this course is of the inelastic, post-yielding type. Brittle fracture and nonlinear elasticity are not included.

Review

Structural nonlinearity

Structural nonlinearity can be defined as a structural system that results in having stiffness analysis components that are not constant.

A structural system is nonlinear if an accurate stiffness matrix or load vector for use in a linear stiffness analysis would contain expressions instead of numerical values, and the expressions include variable(s) such as:

- member/joint deflection, slope, displacement, or rotation
- the location along the length of a member
- direction of force(s)
- extent of strain
- magnitude of reaction
- time

A structural system with one or more instances of acting/engaged structural nonlinearity cannot be analyzed by only a single linear stiffness analysis

The sources of structural nonlinearity can be summarized as:

Geometric nonlinearity	Contact	Time
Material nonlinearity	Shape	

All nonlinearities that are possible in a structural system should be considered. Consider whether the nonlinearity is present, if it will help emulate real-world behavior, and if it will be non-negligible, influential, and consequential. If consideration leads to including nonlinearity in analyses then the analysis method chosen will need to be able to solve that type of nonlinearity and any other nonlinearities that are concurrently in the same structural system. Concurrent nonlinearities must be analyzed concurrently as nonlinear results cannot be combined or superimposed, and nonlinear results cannot be extrapolated or scaled.

Nonlinear analysis methods

A single linear analysis (1st-order analysis) cannot typically produce nonlinear results. The following are the analysis methods that can be used to analyze a nonlinear structural system:

Discretization	A linear approximation used to analyze nonlinearity by shape by subdividing the nonlinear members into many sub-members.
Geometric Stiffness Matrix	A linear approximation used alongside a linear analysis to analyze geometrically nonlinear P-delta effects.
A few linear analyses	A rudimentary analysis where the user conducts a few linear analyses and manually adjusts inputs in between analysis iterations. • Used to analyze up to a few phase-type nonlinearities
2nd-order analysis	An aided iterative analysis repeating linear analyses and automatically changing input between iterations. • Used to analyze continuum-type nonlinearities, but cannot analyze geometrically nonlinear large deflections or highly nonlinear, flexible systems
Simulation	An aided 2nd-order analysis that can also slowly, incrementally ramp-up loads to full magnitude. • Used to analyze the same type of nonlinearities as 2nd-order, but can accommodate more movement/flexibility
3rd-order analysis	An aided 2nd-order analysis that can ramp-up loads like simulation analyses, but also uses “large deformations” nonlinear beam mechanics in its analysis iterations instead of the small-deflections linear analyses used by 2nd-order and simulation.

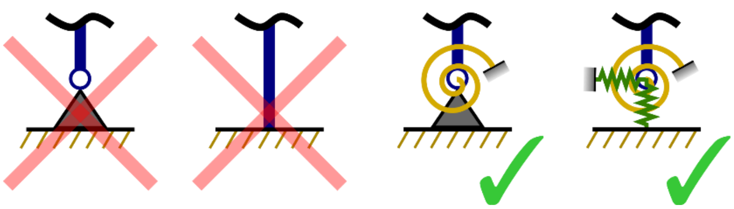
Note that nonlinear structural analyses will not produce misleading or erroneous results if the nonlinearities don’t occur, aren’t engaged, or are found to be inconsequential. This is reassuring when weighing whether to perform nonlinear analyses, and it also allows a nonlinear analysis method to be partially validated by using it to analyze a linear structural system and then comparing to separate linear analysis results.

How can nonlinearity be analyzed?

Idealization

The reader should be accustomed to idealizing structural systems for linear analyses before idealizing structures for nonlinear analyses. Many idealizing assumptions that are used for linear analyses become unsuitable or improper if used for nonlinear analyses. The purpose of idealization is to interpret a structural system in a way that an analysis method can be used to obtain acceptably valid depictions of real-world response. Idealizations made for nonlinear analyses must typically be less bold and more calculated than those made for linear analyses since nonlinear analyses have the tendency to magnify effects to structural response.

Idealizing joints such as column bases as being wholly pinned or fixed for rotations (moments) can be hasty for nonlinear analyses. Most joints and connections will have and should be modeled as having partial fixity, whether elastic or as a nonlinear spring.



3D structures can often be compressed into several planar, 2D models to be linearly analyzed but doing so with nonlinear analyses usually leads to a consequential loss in response accuracy. Any asymmetrical framing or eccentrically applied lateral load will engage 3D response and nonlinear analyses can magnify or exacerbate that response. Most 3D structures should be modeled and nonlinearly analyzed as 3D, though 2D models can be used for checking results.

Other idealization recommendations for nonlinear analyses:

- Foundations sometimes need to be modeled along with the superstructure if their stiffness or attachment could affect force distribution or structural response
- Supporting soils for foundations should typically be modeled using translational and rotational springs

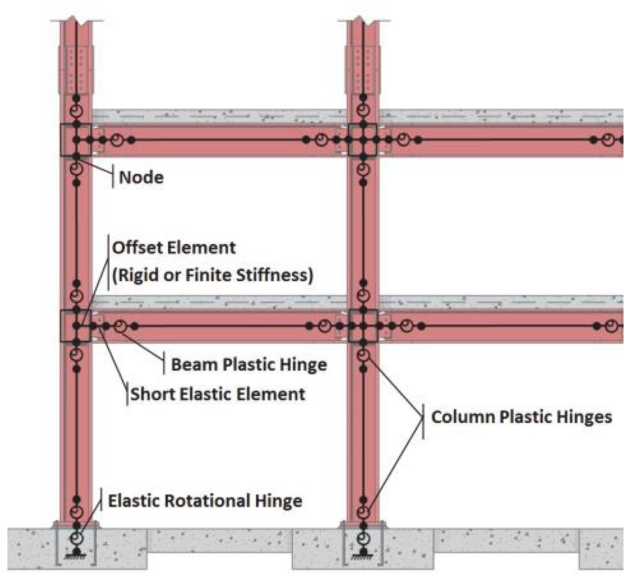


Image A: Illustrative analysis model

- Substructures that are often ignored should be included in the structural model if they present non-negligible stiffness or strength. Examples include stairs and nonstructural walls.

Which loads to ultimately use?

Ultimate or service loads? LRFD (load resistance factor design) or ASD (allowable stress design)?

LRFD vs. ASD

One of the shared principles of the ultimate/LRFD and service/ASD duality of load combinations is that overload conditions can and do occur, and nonlinear analyses are predominantly performed because of P-delta effects that will present some response magnification as compared to a linear analysis of the structural system without the nonlinearity. Nonlinear analysis results cannot be extrapolated or scaled so for a nonlinear analysis to capture the full response magnification associated with overloading it must be run using ultimate/LRFD level loads if the results are used to evaluate strength or stability. After completing nonlinear analyses using ultimate/LRFD loads the results could then be scaled down to service/ASD level for use with the ASD approach of member design, but the magnification does not get scaled.

Quick example

Compare the following linearly-obtained column moments to forces obtained with nonlinear analyses:

$M_{max} = D + 0.6W = 28$, $M_u = 1.2D + 1.0W = 42$, $42/28 = 1.50$

A nonlinear analysis is run using M_{max} and corresponding service/ASD level axial compression P_{max}, and the magnification in column moments is found to be 3% (1.03 times the bending moment from a linear analysis).

A nonlinear analysis is run using M_u and P_u, and the magnification in column moments is found to be 7%. → $M_{u,nl} = 1.07(1.50M_{linear}) = 1.61M_{linear}$

$M_{max,nl\text{-}asd} = 1.03M_{linear}$ → not typically permitted

$M_{max,nl\text{-}lrfd} = 1.61M_{linear} / 1.50 = 1.07M_{linear}$ → as in standards

Sometimes phase-type nonlinearities like directional members can be nonlinearly analyzed using either service/ASD or ultimate/LRFD loads without any discernable difference in the results (aside from proportionally relative load and force magnitudes). However, phase-type nonlinearities with variable thresholds like gaps and friction can pose a situation where at least one of the nonlinearity instances is not triggered by service/ASD level loads but would be by ultimate/LRFD loads. To check for this condition, analyses using ultimate/LRFD loads would also have to be run.

<u>The nature of nonlinearities requires that nonlinear analyses generally be run using ultimate/LRFD level loads and published standards typically require it.</u> Furthermore, the LRFD design/evaluation approach naturally follows for use in member design and evaluation since no post-analysis factoring of forces is required.

<u>Analysis guidance by type of nonlinearity</u>

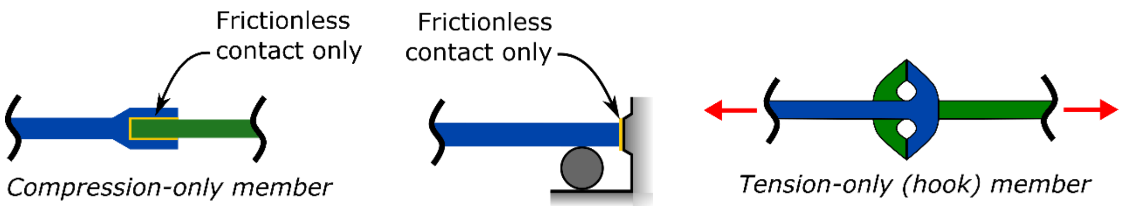
The following figures summarize how several types of nonlinearity can be analyzed. Note that the next course in this series focuses on geometric nonlinearity where more guidance will be provided on analyzing geometric nonlinearity.

Boundary conditions & directional members

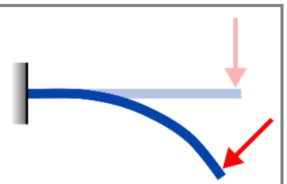
Rudimentary analyses: Iterative presumptive/back-check analyses

Aided analyses: Advanced structural analysis software (SAS) should include most of the possible nonlinear boundary conditions. Nonuniform springs are not as widely available as contact and gap-type supports.

Consequences of ignoring: Inaccurate, misleading, or erroneous member forces or structural behavior



Forces that change direction are rarely encountered in structural engineering and require substantial derivations or full-featured simulation, 3rd-order, or multi-physics software to analyze.

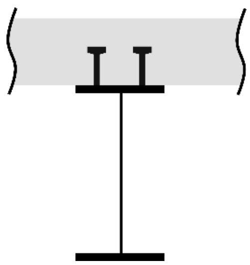


Time

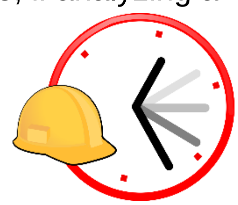
Rudimentary analyses: Sequential analyses using superposition

Aided analyses: Sequential/phased analysis in advanced SAS or simulation, purpose-built software such as for prestressed concrete.

Changing the stiffness of a structure over time is not a widespread feature in analysis software but there are packages available that will allow for members



to be activated and deactivated at specified points in time or will allow for changes in material or cross-sectional stiffness over time. Note that for staged/sequential analyses the way that software accommodates structural distortion must be considered; i.e. as the structure moves how are changes and additions accommodated? For example, if analyzing a 50-story building while construction is in-progress at the 30th level, do the joints of the future 40th level translate downward vertically as the lower levels compress elastically, or do future columns get "stretched" upon activation to conform to those unmoved joints?



Consequences of ignoring: Wholly inaccurate or grossly under/overestimated member forces and stresses. Improper force and stress distribution within composite members.

Notes: Any analyses that consider time must track the force/stress states of every joint, member, and several locations within many members at each point in time that is considered. Sequential construction analyses and even kinematics where time is used as a pseudo-variable can and will present cases where the extrema for certain responses occur in other phases or points in time than the final phase or condition.

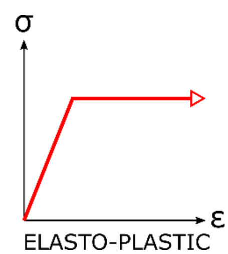
Material nonlinearity

Rudimentary analyses: Elasto-plastic material models, plastic analyses that identify hinge locations and collapse mechanisms, Wood (Rankine-Merchant) method

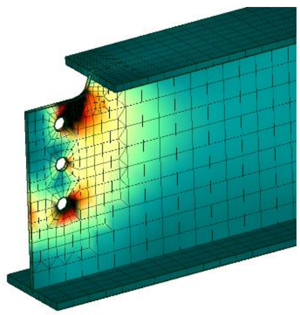
Aided analyses: Pushover analysis in advanced SAS, inelastic analysis in some advanced SAS and more advanced software

Consequences of ignoring: Reduced structural capacity, reduced energy dissipating for seismic, blast, and other energy resisting applications, wrong collapse behavior/capacity, erroneous distortions

Notes:

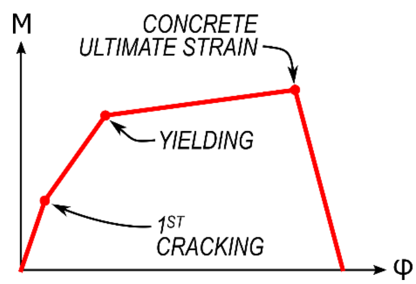


1D-line elements are sometimes used in materially nonlinear analyses with an elasto-plastic stress-strain relationship and 1D-line elements cannot capture the local effects that are commonplace in inelastic behavior. 2D-area or 3D-volume elements should be used to explicitly analyze post-yielding behavior, or to analyze the onset of yielding using a linear elastic analysis when local effects will be influential (stiffeners, holes, irregular geometry).

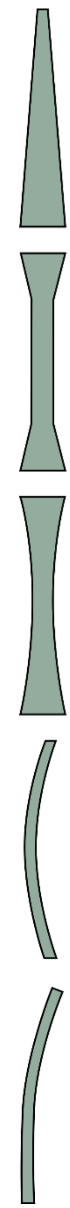


Inelastic analyses should include the effects of member stiffness loss due to inelasticity, residual stresses that may initially be present in members, initial geometric member imperfections, and geometric nonlinearity. High ductility will likely be a goal and requirement of any inelastic analysis used for design purposes, which entails ensuring no member instabilities (crippling, lateral torsional buckling, etc.).

Software packages where users can input custom M-phi (moment-curvature) backbone curves or packages that have inherent concrete capabilities can determine the extent of cracking flexural concrete members and determine the effect of the cracking on stiffness.



Nonlinearity by shape



Rudimentary analyses: Tabulated factors for some applications, and some closed-form solutions for specific cases

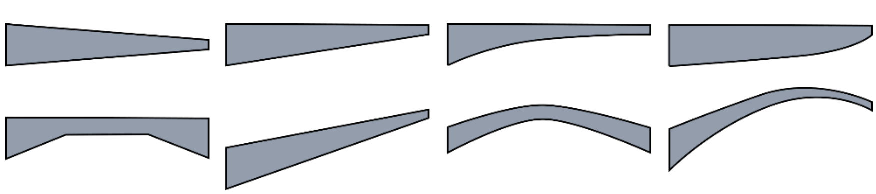
Aided analyses: Discretization of those members into many sequential sub-members, SAS that can accommodate curved members or functions in place of constant member properties, finite element modeling using 2D-area or 3D-volume elements

Consequences of ignoring: Improper distribution of forces within indeterminate structures (non-prismatic members), missing member forces such as torsion in curved members or shear in haunched members, inaccurate movement results (deflections, displacements, etc.)

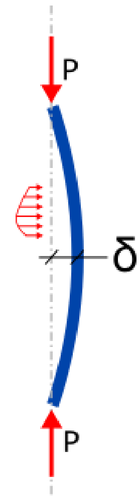
Curved Members notes:
Curved members with more than slight curvature can have nonlinear bending due to the neutral axis being located between centroidal axis and the center of curvature.

Take care with curved member end restraints/releases considering that longitudinal axial tension or compressive thrust is expected in gravity-loaded members that curve in elevation. Torsion, torsional warping, and restraint of warping must be considered for gravity-loaded members that curve in plan. AASHTO recognizes horizontal curved girder-to-girder braces as primary members since they distribute torsional movement to multiple girders and resolve equilibrium, unless the girders are closed shapes. Refer to AISC Design Guide 33: Curved Member Design and AASHTO 4.6.1.2 for additional guidance.

Tapered Members notes:
"Typically, eight sub-members per span will give sufficient accuracy for… a beam loaded statically with cross-sectional properties that vary smoothly."[4] Initial out-of-straightness (P-δ) may need to be considered for welded metal members due to fabrication distortion. Additionally, P-Δ effects will likely need to be considered in the analysis of tapered members used in structural frames. Refer to AISC Design Guide 25: Frame Design Using Web-Tapered Members for additional guidance.

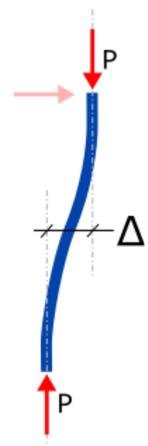


Geometric nonlinearity – small deflections



Rudimentary analyses: Many P-delta situations can be analyzed using a manually navigated 2nd-order analysis, though this could be quite tedious for multi-member frames. Both AISC[5] and ACI[7] have traditionally provided alternate moment magnification methods with linear analysis results to substitute for a 2nd-order nonlinear analysis. A structure must qualify to use the alternate methods by essentially being laterally stiff enough to preclude large magnifications.

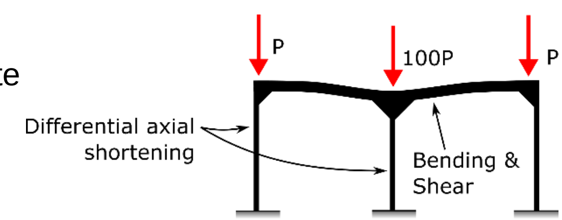
Aided analyses: Many SAS packages only analyze for P-δ and P-Δ effects; determine if other geometric nonlinearity (GN) effects need to be analyzed (axial deformations, snap-through, etc). To use SAS to analyze P-δ and P-Δ effects, first assess and judge:



- Whether the software solves by approximation or by 2nd-order analysis
- Appropriate discretization of members subject to axial compression and P-δ effects; a general recommendation suggests "two to six elements depending on desired accuracy"[8]
- Which displacements are considered for P-Δ effects; some SAS uses only the lateral drift in each of the two principal directions, others use the vector combination of lateral drifts, and some also include the translations from global torsion

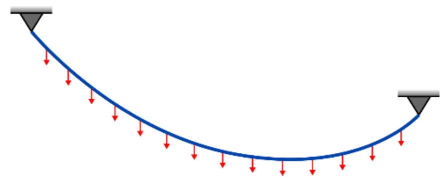
Consequences of ignoring: Underestimated internal forces and structural movement, overestimated structural stability. Will generally result in having a structure or members that are more prone to buckling instability than indicated with a linear analysis.

Note: Only some SAS can accommodate axial shortening of members, and it is typically computationally tedious.

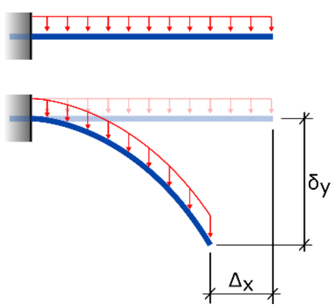


Geometric nonlinearity – moderately large and large deflections

Cables: Tension-only cable (or chain, rope, wire, etc) members with negligible bending stiffness can only be linearly analyzed when cables are longitudinally tensioned over short distances without transverse loads. The behavior of transversely loaded cables is highly nonlinear even without considering cable elongation, temperature differential length changes, non-rigid supports, moving loads, etc. Some approximate analysis methods are available for some loadings but software with nonlinear cable analyses capabilities should be used otherwise.



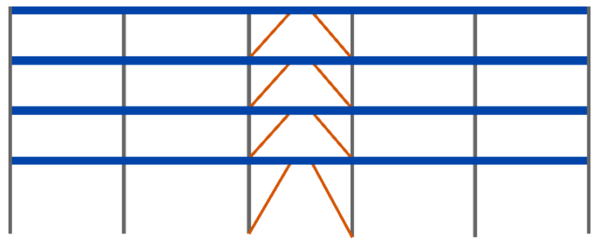
Flexible members: As mentioned in the previous course, once structural movement has exceeded the limits of *small deflection* several implications emerge that significantly increases the analysis complexity. Just beyond the bounds of small deflection a few effects of large deflection (LD) develop, though some linear and approximate methods can still be used with reasonable accuracy. This part of the GN spectrum is often called *moderately large deflections* (MLD). Then, again, as structural movement continues there is another threshold where even those approximate methods are no longer viable as MLD gives way to LD.



Currently, structures exhibiting large structural movements (LD) should almost universally be analyzed using 3rd-order or multi-physics software. MLD can be analyzed with rudimentary methods and will be covered in the next course.

Limits of nonlinear static analyses

Nonlinear *static* analyses can be used to analyze many nonlinear structural systems but should be limited to only providing checking of nonlinear *dynamic* analysis results when the dynamic response is of concern. Specifically, any situation where dynamic loads are applied, when structural systems exhibit multi-mode dynamic response (not a low-rise structure with dominant first-mode response), when cyclic behavior and/or degradation of components is expected, when rate-dependent effects (viscous dampers) are present, when the structure is to use base isolation, etc. NIST's illuminating guide on nonlinear analyses[8] offers many thresholds for transitioning to nonlinear dynamic analyses that were listed above and mentions that a 4-story building structure can be used a rough limit of the applicability of nonlinear static analyses (for seismic).



Standards

Several of the material standards that are referenced by current building codes provide guidance on analysis and may also provide alternate approximate methods to use in lieu of 2nd-order analyses for geometric nonlinearity. Most standards either require 2nd-order analysis for structures that can exhibit non-negligible geometric nonlinearity, or they require that the structure be overdesigned or stiffened until the effects of geometric nonlinearity fade. Standards also provide helpful guidance and insight into other types of nonlinearity. Below are some of the sections of the (U.S.A.) standards that provide guidance and methodology for nonlinear analyses. These references are provided for informational purposes and will not be included in the quiz.

Standard	Section(s)	Comments
AASHTO	4.5 Mathematical Modeling 4.5.2 Structural Material Behavior 4.5.3 Geometry 4.5.5 Discretization for nonprismatic 4.6.1.2 Structures Curved in Plan 5.14.2 Segmental (Concrete) Construction 6.10.1.4 Variable Web Depth (steel)	Large Deflection theory is briefly discussed, there is considerable treatment of members and structural systems that are curved in plan
ACI 318-19	Chapter 6, Structural Analysis 6.7 2nd-order analysis 6.8 Inelastic analysis 11.8.3 Alternative slender wall method	7.7.3.6 requires special consideration of reinforcement anchorage in tapered members
ACI 530-11	3.2.2.4 P-Delta effects	

ADM-10	Chapter C, Design for Stability Geometric nonlinearity is of special concern for aluminum since it is approximately 1/3rd the stiffness of steel	
AISC 360-16	Chapter C, Design for Stability Appendix 1, Design by Advanced Analysis Appendix 7, Alternative Methods of Design for Stability Appendix 8, Approximate 2nd-order Analysis	Design Guides 25 and 33 for Web-Tapered Members and for Curved Members are useful separate documents
AISI S100-16	Chapter C, Design for Stability Inelastic analyses are used extensively in the development of the published strength limit states for cold-formed steel members and connections	
ASCE 19-10	3.4 Structural analysis (cables)	
NDS	3.7.2 Tapered Columns 5.3 (tapered and curved members) 5.4 Special Design Considerations (tapered and curved members)	NDS does not mention geometric nonlinearity despite engineered wood products seeing increased use in tall building structural systems.

Recommendations for nonlinear analyses

Nonlinear analyses can range from being relatively simple to requiring significant preparation and background (linear) analyses just to be able to idealize a structural system prior to nonlinear analysis. Furthermore, seemingly minor attributes can have significant impact on nonlinear results, so it is more difficult to know when an analysis has succeeded and valid results have been obtained.

Structural model features and complexities that are easily accommodated in linear models can have extensive impact on nonlinear analyses. Irregular and asymmetric framing, inclined columns, diaphragm flexibility, eccentricity of diaphragms, openings in members and diaphragms, unbalanced soil retaining forces, and many other conditions can significantly affect the complexity and interpretation of nonlinear analyses and results.

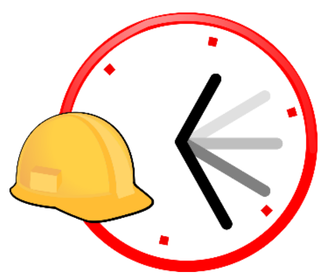
It is recommended that an engineer have considerable experience in linear structural analyses and the process of refining structural models to obtain results that best emulate reality prior to performing complex nonlinear analyses.

> "Computers give the answer to a specific question. Ensuring that questions is the correct one, i.e., that the model represents reality is the job of the engineer." — Niall Macalevey[13], 2010

General nonlinear analysis process:

1. Always start with a linear analysis of the structure. Regardless of the extent of nonlinearity the linear analysis will help validate your idealized structure. Distorted structure diagrams and/or the first several modes shapes can help diagnose errors and improper behavior. If there are issues that remain after adding nonlinearity it will likely be much more difficult to diagnose and identify the problem(s).
2. Use validated nonlinear analysis methods
3. Add nonlinearities gradually if possible; add P-delta effects first, then axial deformations, then material nonlinearity and/or large deformations. Focus on developing a structural analysis model that functions correctly, successfully analyzes/converges, and displays proper response before adding all substructures, minor and incidental loads, load cases, etc.
4. Refine model and address errors/issues, finalize model
5. Check nonlinear results, check assumptions and validate idealizations

Track all relevant response results for any analysis that considers time, including kinematic analyses of phase-type nonlinearities. Consider that the maximum (or minimum) response for a given component, e.g. axial tension in a column, may not occur at the final phase in time. Furthermore, it is often necessary to also track responses directionally, such as tracking negative and positive bending separately.



- Consider graphing the results for key and critical responses versus time if there are three or more time states/phases.

Inelastic analyses are typically complex and time-consuming, with results that are difficult to interpret and validate. Parameters and attributes that are normally glossed over in linear analyses such as member fabrication tolerances and residual internal stresses can have significant impact on results. Most if not all joints, supports, and

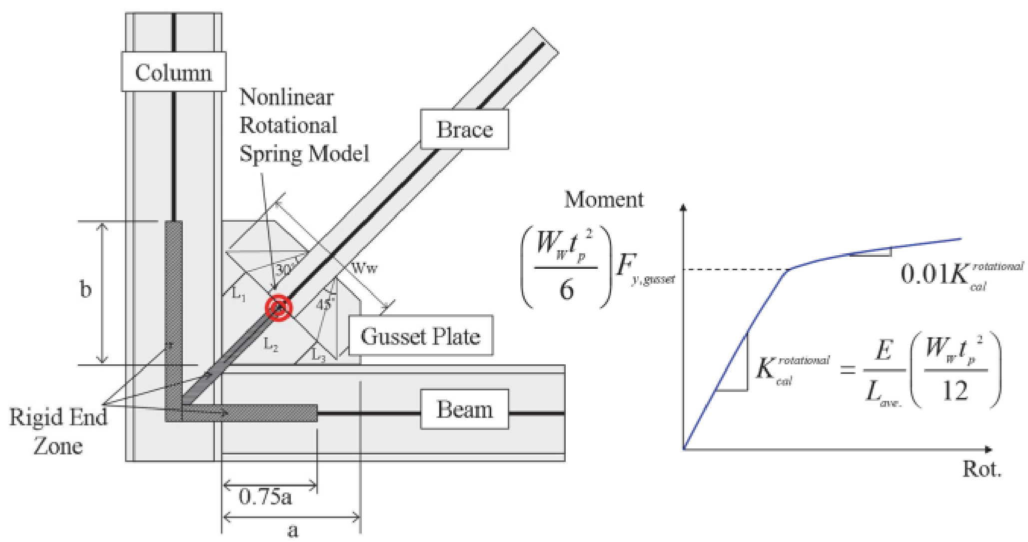
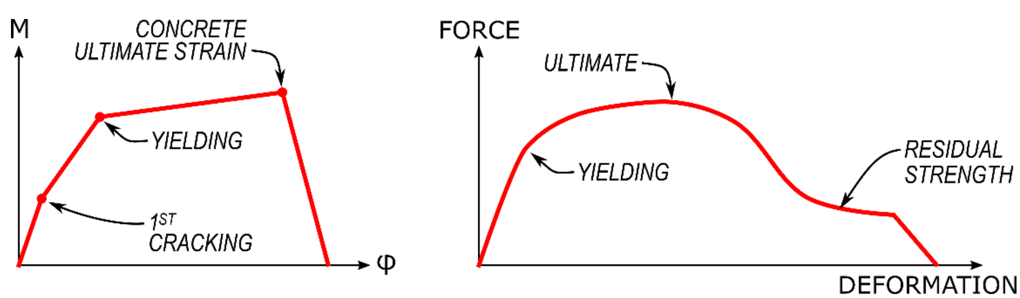


Image B: Illustrative model of beam-column-brace

member connections should be modelled as elastic or nonlinear springs, substantial insight into the behavior of members from an unstressed state to rupture (backbone curves, shown below) is crucial, the use of 2D-area and/or 3D-volume elements in place of some or all 1D-linear (beam or truss) members will be warranted, and knowing how to transition between 1D-line and other types of elements is key. Anisotropic materials add considerable complexity and can introduce discontinuities through parameters like modulus of elasticity for cracking concrete members, and fastener slip for nailed structural wood panels.



Road sign

Acknowledge ignorance and uncertainty and remain open to changing and rerunning the analysis. The results of nonlinear analyses are not predictable so determining how to model a structural system to find worst cases or the most significant response will not be obvious. Well-developed linear analysis models typically require running and rerunning the analysis many times while adjusting and refining the model between; nonlinear analyses will typically require more refining and adjustment along with more instances of rerunning the analysis.

- Member fabrication and erection tolerances cannot be ignored. Furthermore, knowledge of which direction to orient the tolerances may have to be determined by separate rational analysis or by running each orientation individually.
- For nonlinear structural systems each load case constitutes the need to run a dedicated nonlinear analysis (though software may sequence them and perform them automatically). Considering this, moving loads could introduce situations where a set of *load locations of concern* needs to be developed based on multiple initial analyses to narrow-down the total number of needed final analysis runs.
- The effects of changing model parameters may not be known, so consider parametrically covering the range of possibilities. For instance, if a set of springs is being used to emulate soil-foundation interaction and there is uncertainty in the spring rate to use, then consider running each extreme and possibly spring rates between the extremes to reveal the spectrum of response.
- Similarly, admit and acknowledge the magnitude of influence for adjustments to the model and analysis inputs. For example, consider forensic, post-yielding, materially nonlinear analyses of a complex steel connection. It would be misleading to show a particular solution that includes substantial influence from member fabrication tolerances that were selectively modeled without also showing the results for models where the tolerances were differently oriented. Do not “cherry-pick” model and analysis parameters to obtain a desired set of results, rather, show that a certain set of results exists within a range of possible results and define the bounds of that range.

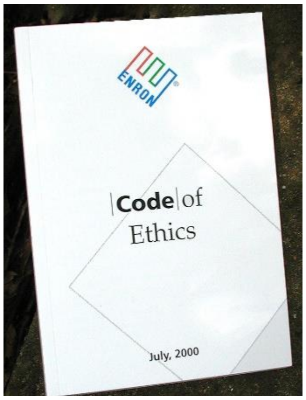


Image C: "Enron Ethics Manual"

Errors and models that diverge or will not converge will be encountered when performing nonlinear analyses with software.

- Check that the selected solver, analysis options, the types of nonlinearities, and other parameters (element types, loading types, etc) are compatible.
- Iterative 2nd-order (or more advanced Simulation and 3rd-order) analyses that will perform many analysis iterations but diverge or will not converge may:
  - contain a member that is near or has exceeded its global buckling capacity or is generally too flexible.
  - sometimes be diagnosed using software tools that indicate what component or part of the structure is experiencing relatively larger structural movement.
  - not have been configured in a way to exit the solver.
- Iterative solvers need to be configured to know when to exit the solving process. If an analysis will not converge then the solver may:
  - need to perform more iterations to converge than the settings currently allow.
  - need a max strain, rupture point, or maximum number of iterations to finish analyzing material nonlinearity.
  - need to use a lower convergence threshold (less precise) for highly flexible structural systems (e.g. cable networks).

Image D: "Basement Exit Sign"

- A structural analysis solver that won't initiate or returns an error on the first couple of iterations could be responding to structural systems that may:
  - contain joints, members, elements, or other components that are moving rigidly (without providing for such movement using stabilization or rigid-body motion flags). Remember that instances of many types of phase-type nonlinearities can cause structural degrees of freedom (DOFs).
  - contain cables or fabric elements without any flexural stiffness that need to be pre-analyzed for prestress prior to analyzing them within the context of a larger structural system. This pre-analysis determines the geometry of the cables/fabric under dead load prior to applying transient loads for the main analysis. Check whether this pre-analysis is automatically performed in the software settings and/or manual.

  - contain other sources of error that similarly affect 1st-order analyses such as using members or elements with improper DOFs, creating coincident joints or members, over-releasing and other internal instabilities, units errors, not specifying the minimum section and material properties necessary for analysis, etc.

- The structural system that was modeled may be too nonlinear and/or complex for the solver or the software (highly nonlinear bridge shown). Try the following as possible, listed in order of preference:
  - use more load ramping steps (simulation or 3rd-order)
  - use a more robust solver, as available
  - remove linear portions of the structure and replace with reaction forces at the points where the removed portions were attached
  - divide the overall structure into separate idealized structures using symmetry, common features, or by specifying physical separations, as possible
  - use coarser density of elements or sub-members (less elements and submembers, but still sufficient for reasonable accuracy)
  - add stiffness to highly flexible parts of the structure, or use stabilization or movement damping features

Image E: Harbor Drive Pedestrian Bridge, San Diego, California

Take great care if using stabilization to allow analysis of a flexible structural system.

- Errors that emerge from rigid-body movement or divergent iterative analyses are warning signs that should not be disregarded; ensure that the implications of such warning signs are heeded prior to using software's stabilization features.
- Soft springs require a thorough understanding of structural behavior to be used to create or stabilize a DOF. The use of soft springs can be dangerous if they artificially mask or hide an instability; knowledge of and prevention of an

instability is a far more desirable outcome than a potentially unstable structure with unconservative design forces.

Validate nonlinear analysis methods and check nonlinear results to provide for quality assurance and quality control.

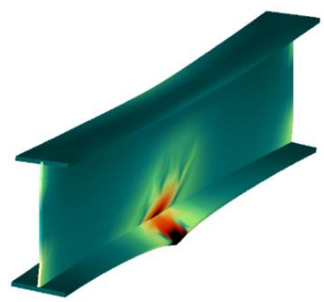
- Do not use nonlinear analysis results from an unproven source unless the results can be otherwise checked.
- Validating nonlinear analysis methods and checking results are separate and both are required. Validating a nonlinear analysis method may only need to be done once but checking results should be done for every analysis that is performed.
- Validate nonlinear analysis methods (software or rudimentary) using published benchmark problems to gauge general accuracy. Then check all results, in order of preference:

Image F: Engineer reviews data

  - Check nonlinear results against an outside source, such as another software package or by retaining third-party engineering review.
  - Replicate and simplify your analysis model, and have another engineer develop that model and run that analysis if possible. Simplify the structural system but retain nonlinearities. Using engineering judgment some complexity can be removed while expecting reasonably similar structural response. Compare the resulting structural response to (ideally, and as available) benchmark problems, tabulated data, worked examples, or with the full, non-simplified model. Exploit the simplifications to help identify any modeling errors and improper response.
  - If no other sources of comparison or verification are available then use care and judgment if using separate analyses of crude linear idealizations to check nonlinear analysis results. Idealize one or more structural systems that are linear but capture, envelope, overestimate, or emulate the expected nonlinear behavior. <u>These linear analyses do not substitute for nonlinear analyses</u>. The errors will not be negligible but there are often one or a few linear idealizations that can provide a few key comparison values to help verify nonlinear analysis results.

Knowledge of the behavior, effects, consequences, and nuances of structural nonlinearity can often be more valuable than detailed knowledge and understanding of theory. Of special importance is **knowing the thresholds of nonlinear behavior**.

- Many codes and standards offer a formula using linearly-obtained drift or deflection that is compared to a provided constant, and if the structure/member is stiff enough a nonlinear analysis is deemed unnecessary. This allows an engineer with an understanding of structure lateral and transverse member behavior to avoid nonlinearity knowingly and safely, by ensuring a structure with enough stiffness to preclude significant influence from P-δ or P-Δ effects.
- The small-deflections assumption used in linear stiffness, 2nd-order, and simulation analyses is valid until member deflections exceed approximately one-half of the member depth, and/or until slopes or rotations exceed approximately ten degrees.
- Determine what depth of results are required for yielding-type material nonlinearity. Some situations only require knowledge of where yielding occurs, and these situations can sometimes be resolved by using the onset of yielding as obtained from a linear (elastic) analysis. Alternatively, if the strength of members and materials cannot be increased to avoid yielding and plastic, post-yielding behavior does need to be analyzed, then inelastic analyses will be required.

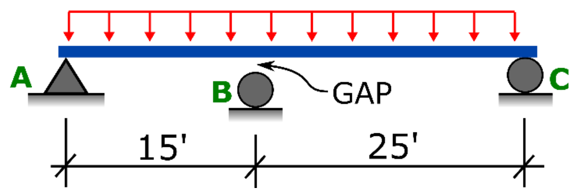


Examples

The following examples and the ways in which they were solved were intentionally crafted to show the nature of structural nonlinearity. The engineering solutions are not shown with the intentions of teaching rigorous calculations methods, but to illustrate the iterative nature of nonlinear analysis, emphasize verification/checking of results, show that rudimentary methods can be fairly accurate, demonstrate approximate methods, and highlight potential pitfalls.

> i.e. Pay more attention to the approaches, results trends, and discussion sections between calculation modules than the minutiae of the calculations.

Example 1 – Gap Support	
Included Nonlinearity	Nonlinear boundary condition; gap-type support
Possible Nonlinearity	1. Geometric nonlinearity; large or moderately large deflections 2. Material nonlinearity; yielding and post-yielding response
Brief	Uniformly loaded, simply-supported beam with a gap support within the span. Phase-type nonlinearity that has two states to be analyzed; the first phase is the closing of the gap with the structural system being a single simply-supported member, and the second phase is the application of any remaining load onto a two-member, continuous beam structural system. Detailed calculations will not be shown since these two phases can each be linearly analyzed.



L = span =	40 ft	12.2 m	4.6 m & 7.6 m between supports
w_u =	3.0 kips/ft	43.8 kN/m	
E =	29,000 ksi	200 GPa	
I =	920 in^4	38,300 cm^4	approx: W12x106 | W310x158
Gap magnitude =	2.00 in	51 mm	including as-measured camber tolerance

Note: the uniform load, w_u, is an ultimate/LRFD load – reminder that nonlinear analyses for strength or stability purposes should be run at the ultimate/LRFD load level

Assume: Geometric and Material Nonlinearity do not occur

Methodology

1. Apply unit uniform load to single member, 40 ft beam (linear analysis #1)
2. Divide the gap magnitude by the deflection result at 15 ft and use that ratio to find w_{gap}, the uniform load magnitude that just closes the gap, then scale the results from the first linear analysis using the same ratio.
3. Run linear analysis #2 on a two-member continuous beam structural system using a uniform load magnitude = w_u - w_{gap}
4. Combine the results of the two linear analyses
5. Check results and assumptions
6. Discuss obtaining design forces from results

Example 1 Uniformly loaded, simply-supported beam with a gap support within the span

#1 & #2 -- First linear analysis and scaling to just close gap

Simply supported beam with open vertical gap over interior support

w_{unit} = 1.00 k/ft = 14.6 kN/m

deflection at 15 ft = -1.999 in = -50.8 mm

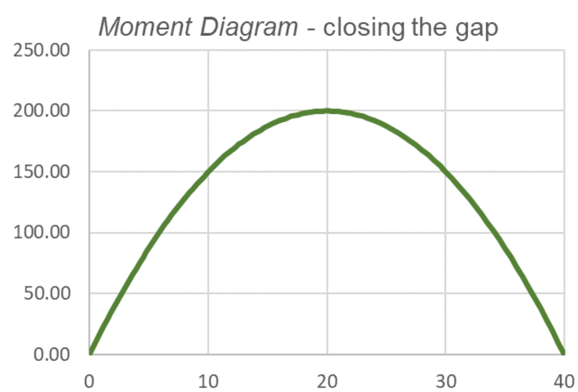
gap / defl @ 15 ft = 1.0005

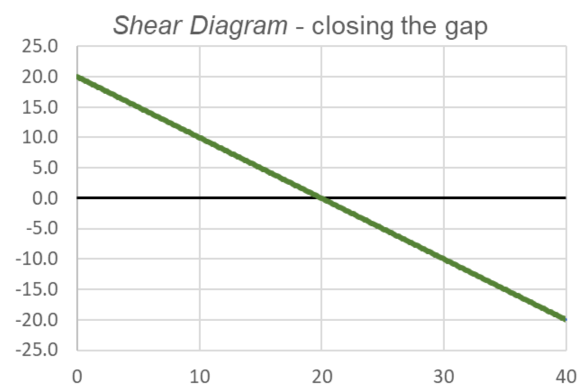
→ Scaling of unit load analysis results using ratio = 2.00/1.999 = 1.0005 ⇒ 1.0

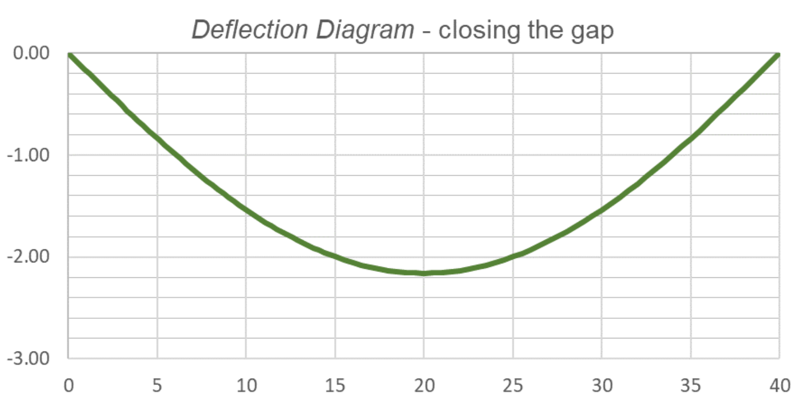
w_{gap} = 1.0 k/ft * 1.0 = 1.00 k/ft = 14.6 kN/m

	k	kN		k-ft	kN-m
$V_0 = R_A = R_C$ =	20	89	M_{15} =	188	254
V_{15} =	5.0	22	M_{28} =	168	228

units in graphs are k-ft, k, and in







Example 1 Uniformly loaded, simply-supported beam with a gap support within the span

#3 – Second linear analysis, after gap has closed

Two-span continuous beam, 1° indeterminate

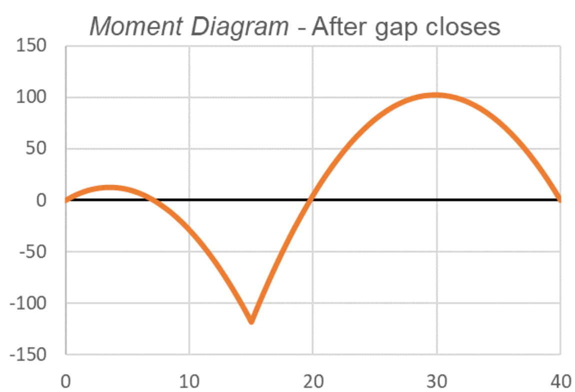
$w_{2span} = w_u - w_{gap} = 2.00$ k/ft = 29.2 kN/m

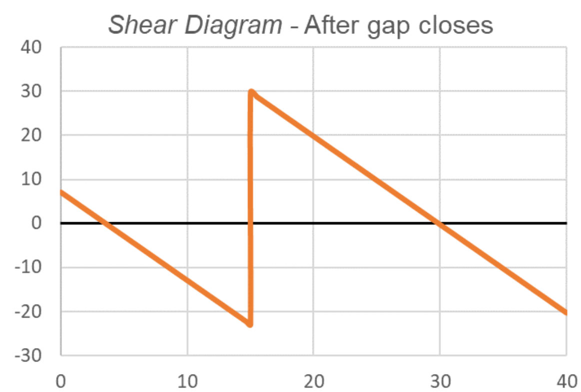
	k	kN
$V_0 = R_A =$	7.08	31.5
$V_{L15} =$	-22.9	-102
$V_{R15} =$	29.8	132
$V_{40} = R_C =$	-20.3	-90.1
$R_B = V_{L15} + V_{R15} =$	52.7	234

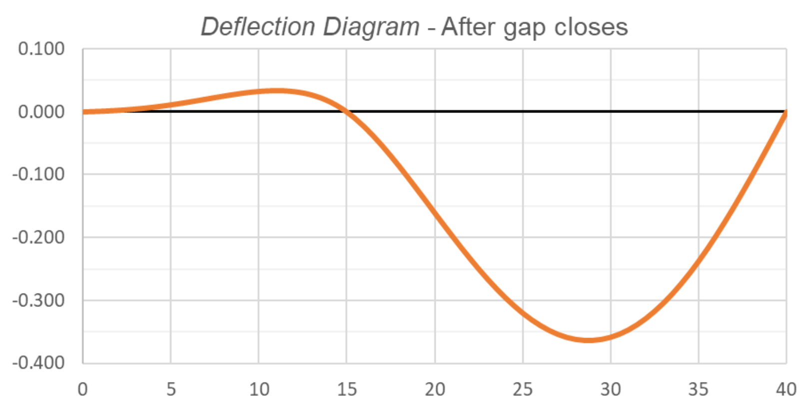
	k-ft	kN-m
$M_{15} =$	-119	-161
$M_{28} =$	99.0	134

	in	mm
$\delta_{15} =$	0.0	0.0
$\delta_{24} =$	-0.295	-7.50

units in graphs are k-ft, k, and in







Example 1 Uniformly loaded, simply-supported beam with a gap support within the span

#4 – combine phase results of two linear analyses

$w_{total} = w_u - w_{gap} = 3.00$ k/ft = 43.77 kN/m

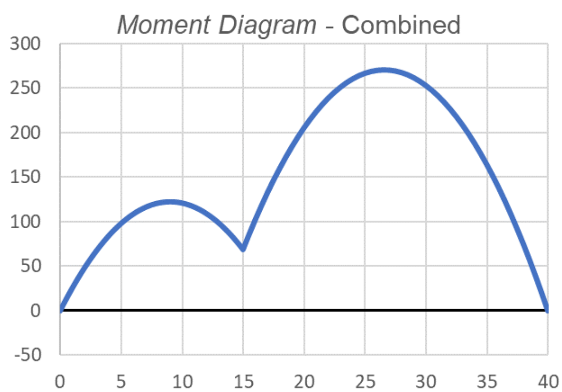
	k	kN
$V_0 = R_A =$	27.1	121
$V_{L15} =$	-17.9	-79.7
$V_{R15} =$	34.8	155
V_{40} $= R_C =$	**40.3**	**179**
R_B $= V_{L15} + V_{R15} =$	**52.7**	**234**

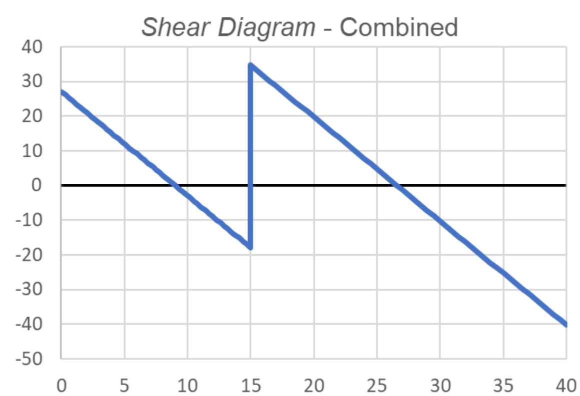
	k-ft	kN-m
$M_{15} =$	68.8	93.2
M_{28} $=$	**267**	**362**

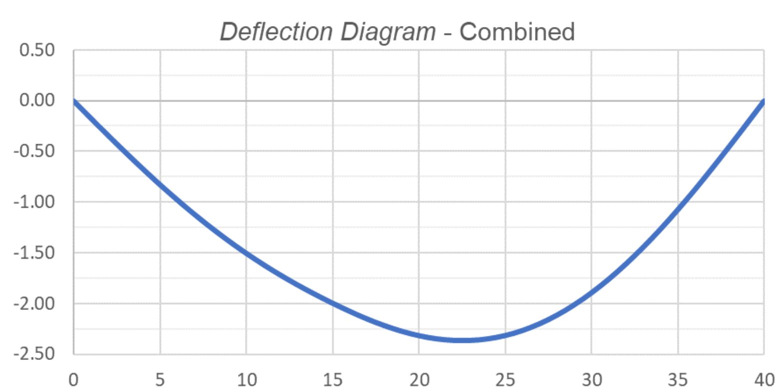
	in	mm
$\delta_{15} =$	-1.999	-50.8
δ_{24} $=$	**-2.351**	**-59.7**

Bold values above are maxima

units in graphs are k-ft, k, and in



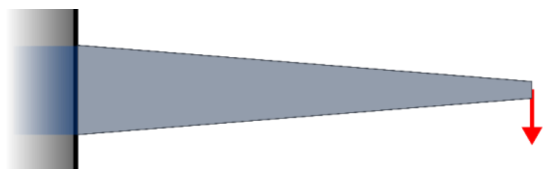




<table>
<tr><td colspan="2">Example 1 Uniformly loaded, simply-supported beam with a gap support within the span</td></tr>
<tr><td colspan="2">#5 – Check results and assumptions</td></tr>
<tr><td>
• check summation of forces in the vertical direction
▪ 3.00 k/ft * 40 ft = 120 k
▪ $R_A + R_B + R_C$ = 27.08 + 52.67 +40.25 = 120.0
▪ 120 = 120, OK
• check material elasticity
▪ assuming the unbraced length does not exceed L_r
▪ S_x (W12x106) = 145 in^3
▪ F_b = M/S = (12)*267 k-ft / 145 in^3 = 22.1 ksi
▪ F_b = 22.1 ksi < F_Y = 50 ksi → elastic
• check beam stability
▪ lateral-torsional buckling involves the member bracing and is beyond the scope of this example
• check validity of assuming no geometric nonlinearity
▪ There are no axial loads in the beam so P-δ effects are precluded
▪ δ_{max} = -2.351 in
▪ approximate deflection needed to transition to moderately large deflections = 1/2 beam depth ≈ 6.0 in
▪ 2.351 < 6.0 → small deflections analysis valid
</td><td>
= 533.8 kN
= 120.5 + 234.3 + 179 = 533.8
533.8 = 533.8, OK
= 152.4 MPa
152.4 MPa < 345 MPa → elastic
= 59.7 mm
= 152 mm
59.7 < 152 → analysis valid
</td></tr>
<tr><td colspan="2">#6 – Discuss obtaining design forces from results</td></tr>
<tr><td colspan="2">
It is important to note that the design forces for a location would be the maximum that the member experienced for each force direction. In this example the open-gap positive moment at the 15 ft (4.6 m) location was large enough that the negative moment from the closed gap support did not reverse the moment direction. If the moment from the end results (closed-gap, combined) were used to design that portion of the member it would be 69/188 = 93/254 = 37% of the maximum flexure that the beam experienced at that location just as the gap closed.

If the gap would have been small it would be important to recognize <u>both</u> the small positive moment that occurred before gap closing at the 15 ft (4.6 m) location, and the large negative moment from the remaining load application. Tracking the envelope of forces directionally can be critical for anisotropic materials like wood and concrete.
</td></tr>
</table>

Example 2 – Discretizing nonprismatic members	
Included Nonlinearity	Nonlinearity by Shape
Possible Nonlinearity	1. Material nonlinearity; cracking and possibly yielding response 2. Geometric nonlinearity; large or moderately large deflections
Brief	A linearly doubly-tapered-depth cantilever beam is loaded at its tip with a transverse concentrated force. Determine the tip deflection of the beam using linear approximation via discretization, and compare the results to an exact, theoretical solution.



L = span =	50.0 ft	15.2 m
h = beam depth varies:	6.0 ft to 1.0 ft (@ tip)	1.8 m to 0.3 m (@ tip)
b = beam uniform width =	2.0 ft	0.6 m
P_u =	100 kips	450 kN
$E_{CONCRETE}$ (4000 psi / 27.6 MPa) =	3,600 ksi	24,800 MPa

Scope: Flexural cracking of the concrete cantilever would constitute material nonlinearity or the nonlinearity of member stiffness that changes over time, as the flexural stiffness EI would need to be reduced to account for the change from I_{gross} to $I_{cracked}$. The intent of this example is to show linear approximation (discretization) of a member that is nonlinear by shape, and the cracking check and nonlinearity associated with cracking is beyond the scope.

Methodology

1. Attempt to find a “shortcut” approximate method by performing a few rudimentary analyses of a simplified member
2. Develop finite element formulation for sub-members, discretize the member into sub-members, calculate section properties for each sub-member, perform linear analysis of sub-members
3. Repeat step #2 using SAS with additional sub-member divisions
4. Compute the exact, theoretical solution and compare with the discretization results, discuss errors
5. Discuss “shortcuts”

Example 2 Tapered cantilever beam discretized into sub-members

#1 – Attempt to find a "shortcut" approximate method

Are there any shortcuts to finding the deflection at the tip of the tapered member? Common ideas are to use the average section properties or to use section properties for a certain point along the length of the member, and then use the formula for prismatic members.

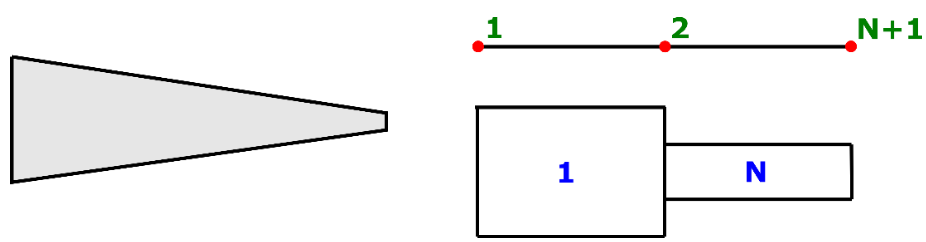
$$\delta_{prismatic} = \frac{PL^3}{3EI_{shortcut}}$$

Try the average of the moment of inertia values of the member endpoints, the average of the values all along the member (using discretized properties from further ahead in the example), and the values at the half-span and one-third-span points:

	I (in^4)	δ_{TIP} (in)	I (m^4)	δ_{TIP} (cm)
$I_{AVERAGE-ENDS}$	374,976	5.34	1.561 x 10^{-1}	13.55
$I_{AVERAGE-ALL}$	226,800	8.82	3.16 x 10^{-2}	22.40
I (@ L/2)	148,176	12.40	6.17 x 10^{-2}	31.51
I (@ L/3)	281,216	7.11	1.17 x 10^{-1}	18.06

These values will be revisited after all other results are computed

#2 – Finite element formulation for sub-members



Note! Remember that it is the cross-sectional dimensions that must be determined for each point along the length. <u>The moment of inertia cannot be determined at the support and the tip and then be linearly scaled for locations between the endpoints</u>, as the moment of inertia is a cubic function of member depth. This is an easy error to make and will be revisited at the end of this example.

# of sub-members = N

# of joints = N + 1

Subscripts: i = counting integer, SM = sub-member, J = joint

$I_{SM,i}$ = sub-member property of sub-member # i = average of $I_{J,i}$ and $I_{J,i+1}$ properties

$I_{J,i}$ = member property at joint # i location

Example 2 Tapered cantilever beam discretized into sub-members

#2 – Finite element formulation for sub-members, discretize the member into sub-members, calculate section properties for each sub-member, perform linear analysis of sub-members

Try an increasing number of discretized sub-members. A system with 2 sub-members will be analyzed, then analyze 4, 8, 20, and 50 sub-member systems using SAS.

Calculate the section properties at the joint locations.

N = 2

x (ft)	h (ft | m)	I_J (in^4)	I_{SM} (in^4)	I_J (m^4)	I_{SM} (m^4)
0	6.0 | 1.83	746,496		3.107×10^{-1}	
(sub-member #1)			447,336		1.862×10^{-1}
L/N = 25	3.5 | 1.07	148,176		6.17×10^{-2}	
(sub-member #2)			75,816		3.156×10^{-2}
L = 50	1.0 | 0.305	3,456		1.44×10^{-3}	

Using the finite element method and for cantilever sub-members, the following are the deflection and slope expressions at a joint:

$$\delta_V = \frac{VL_{SM}^{\ 3}}{3EI_{SM}} \qquad \theta_V = \frac{VL_{SM}^{\ 2}}{2EI_{SM}} \qquad \delta_M = \frac{ML_{SM}^{\ 2}}{2EI_{SM}} \qquad \theta_M = \frac{ML_{SM}}{EI_{SM}}$$

$$V = P_u \qquad M = P_u\left(L - \frac{L(i-1)}{N}\right)$$

Derive the joint deflection for the ith joint:

$$\delta_{J,i} = \sum \delta_V + \sum \delta_M + \sum L_{SM,i-1}\tan(\theta_{V,i-1} + \theta_{M,i-1})$$

i	$\delta_{V,i}$ (in)	$\Theta_{V,i}$ (rad)	$\delta_{M,i}$ (in)	$\Theta_{M,i}$ (rad)	$\delta_{V,i}$ (cm)	$\delta_{M,i}$ (cm)
1	0	0	0	0	0	0
2	0.559	2.794×10^{-3}	0.838	5.589×10^{-3}	1.42	2.13
3	3.297	1.649×10^{-2}	0	0	8.37	0

The deflection at the tip of the 2 sub-member system would then be:

(0.559 + 3.297 + 0.838) + (25*12)tan(2.794×10^{-3} + 5.589×10^{-3}) =

4.694 + 300(0.00838) **= 7.209 in = 18.31 cm**

Example 2	Tapered cantilever beam discretized into sub-members

#3 – Repeat step #2 using SAS with additional sub-member divisions

Use SAS to perform linear stiffness analysis of discretized member systems using 2-node beam elements. Reanalyze the 2 sub-member system that was used in the rudimentary analysis on the previous page, then analyze 4, 8, 20, and 50 sub-member systems. The sub-member section properties (area and moment of inertia) have been calculated in the same manner shown above with average properties of each sub-member's end-joints and are not included here for brevity.

N # of sub-members	δ_Y (in)	δ_Y (cm)	
2	7.207	18.31	**← verifies the rudimentary solution above**
4	8.158	20.72	
8	8.415	21.37	
20	8.478	21.53	
50	8.487	21.56	

#4 – Compute the exact, theoretical solution

Wong, Gunawan, et al[18] published the following expression for the tip deflection of a linearly doubly-tapered-depth cantilever. Note that the origin occurs at the tip (free end) of the cantilever so that x = L at the fixed support end of member.

$$\delta_{TIP} = \frac{P}{2a^2 EI_O}\left(\frac{2}{a}\ln\alpha - \frac{L(3\alpha-1)}{\alpha^2}\right)$$

where:

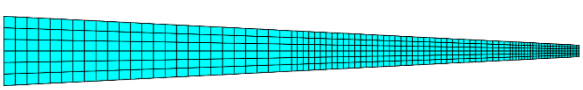
$$\alpha = \frac{h(L)}{h(0)}$$

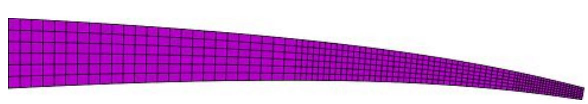
$$a = \frac{\alpha - 1}{L}$$

For this example: $\alpha = 6.0,\ a = 8.33\times 10^{-3},\ I_O = 3456\ \text{in}^4\ \ (143850\ \text{cm}^4)$

$\boldsymbol{\delta_{TIP} = 8.489\ \text{in} \quad = \quad 21.56\ \text{cm}}$

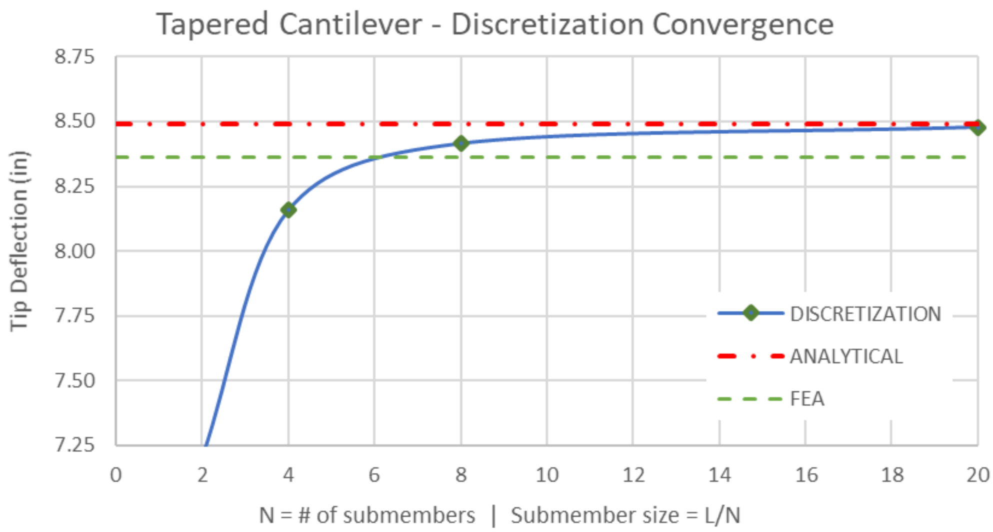
For comparison, a FE model was developed using 3D-volume, 8-node "brick" isoparametric linear solid elements. A total of 2,088 elements and 3,080 joints were modeled, and the linear stiffness analysis resulted in a tip deflection of 8.362 inches (21.24 cm). This is 1.5% less than the exact solution, and FEA commonly shows stiffness increases of up to a few percent over theoretical stiffness.





Example 2 Tapered cantilever beam discretized into sub-members
#4 – Compare theoretical result with the discretization results, discuss errors

The graph below shows 3 of 5 of the discretization data points for the vertical tip deflection, with a trendline using all 5 data points to help show convergence. The red dashed line indicates the theoretically exact solution.



The error when using 4 sub-members is only 4% and the error with 8 sub-members as recommended by AASHTO is less than 1%. The next course will discuss how more sub-members are required when analyzing geometric nonlinearity or stability in nonprismatic members. With modern structural analysis software and even with spreadsheets it is best to use more sub-members as the modeling/analysis time required to increase the quantity of sub-members is negligible.

What if the moment of inertia values had been linearly scaled from the moment of inertia values for the two member endpoints (fixed support, and free tip)? This is the common pitfall that engineers make when discretizing tapered members and was mentioned earlier.

If nonlinear section properties (units^3, units^4, etc) are linearly scaled the convergence graph looks exactly as above, except the discretization results converge on the wrong value. For this example, linearly scaling the moment of inertia results in converging on a tip deflection value of 4.00 inches (10.2 cm) → **112% error**

This potential pitfall illustrates the need to check and verify analysis methods:

- Hand-verify section properties for a few interior locations along a nonprismatic, discretized member
- Verify overall analysis against benchmark solution, other analysis methods, etc.

Example 2 Tapered cantilever beam discretized into sub-members

#5 – Discuss "shortcuts"

Revisiting the "shortcut" results from the 1st step, above, that were obtained using a simplified section property with the deflection formula for prismatic members:

Case	δ_Y (in)	error	δ_Y (cm)	NOTE
$I_{AVERAGE\text{-}ENDS}$	5.34	59%	13.55	
$I_{AVERAGE\text{-}ALL}$	8.82	4%	22.40	1
I (@ L/2)	12.40	46%	31.51	
I (@ L/3)	7.11	19%	18.06	2

Nonlinearity does not often conform to or exhibit behavior that can be described with a rule-of-thumb. The notes below discuss details of why a "shortcut" is not readily available for this situation, and this is only one of many geometry profiles of tapered nonprismatic members. Members can taper in width, in width and depth, taper only on one face (singly tapered), and can have curved taper profiles, etc.

NOTES:

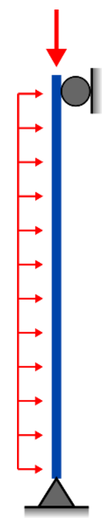
1. This tip deflection was computed using the average of all of the individually computed section properties for each joint on the 50-sub-member discretized system. It is not surprising that the tip deflection computed using this average is fairly accurate, but it may be surprising to learn that <u>the accuracy is just a coincidence</u>. Nonlinearity is not predictable. Rerunning this example with a range of taper ratios (this example used 6:1) shows that a linear taper ratio of 6.66:1 will have the deflection that is computed using the prismatic cantilever deflection formula ($PL^3/3EI$) using $I_{AVERAGE\text{-}ALL}$ match the exact solution. Otherwise, the error ranges up to 25% for other taper ratios.

2. This tip deflection was computed using a single member, prismatic cantilever that had the section properties equal to the tapered member at L/3 from the fixed support. It seems that this approach could be used as an approximate method for rough, first-pass calculations for linearly tapered-depth cantilevers, but that relationship is also nonlinear with respect to taper ratio.

   By back calculating, the moment of inertia at 0.383L on the tapered member is found to provide results that match the exact, theoretical solution when using the prismatic cantilever deflection formula ($PL^3/3EI$) <u>for this example</u>. By repeating the problem with varying rates of taper (1.5:1 up to 10:1) this "shortcut" location to use the moment of inertia from moves from 0.28L to 0.42L.

Example 3 – Column P-δ	
Included Nonlinearity	Geometric; axial compression + member flexure (P-δ) Nonlinearity by Shape (inherent in GN)
Possible Nonlinearity	1. Geometric nonlinearity; large or moderately large deflections 2. Material nonlinearity; yielding and post-yielding response
Brief	A simply-supported column is subjected to concurrent axial compression and member transverse flexure. A 2nd-order analysis is required to include the P-δ geometric nonlinearity. Problem from: AISC 360 Commentary C2.1, Case 1 (pg 16.1-291 in 15th ed) Similarly: ACI 318-19, 11.8.3 compressive P-δ for wall panels

H = span =	28.0 ft	8.53 m
w_u =	0.20 kips/ft	2.92 kN/m
P_u = axial compression =	150, 300, 450 kips	667, 1334, 2002 kN
E =	29,000 ksi	200 GPa
I (W14x48 | W360x72) =	484 in^4	20146 cm^4



Neglect: Shear deformations, camber/sweep fabrication tolerance

Assume: Large Deflections and Material Nonlinearity do not occur

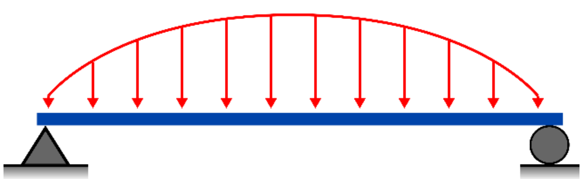
Methodology

1. Perform initial linear, small-deflection analysis and obtain mid-height (MH) lateral deflection and MH bending moment
2. Discuss member nonlinearity by shape for when member is held in a distorted position
3. Use the deflection value from the previous step and calculate the P-δ moment, $M_{P\text{-}\delta 1}$, at mid-height due to the axial compression (150 kips). Determine the additional lateral deflection that would occur due to the calculated P-δ moment. $M_{u,new}$ is then set equal to the moment calculated during the previous step added to $M_{P\text{-}\delta 1}$
4. Repeat step #2 using the $\delta_{P\text{-}\delta,MH,i}$ value from the previous iteration until, at *i* = iteration # = N, the change in moment ($M_{P\text{-}\delta,N}/M_{u,new,N}$) is less than 0.50%
5. Repeat steps #2 and #3 with the two (2) remaining P_u axial compression magnitudes
6. Check results, try approximate method, and check assumptions

Example 3 Column P-δ	
#1 – Perform initial linear, small-deflection analysis (1st analysis step in 2nd-order analysis)	
$w_u = 0.20 / 12 = 0.0167$ kli	$= 2.92$ kN/m
$M_{u,MH} = w_u L^2/8 = 235.2$ k-in	$= 26.574$ N-m
$P_{cr} = \pi^2 EI/L^2 = 1227$ k	$= 5458$ kN
$\delta_{MH,linear} = 5w_u L^4/384EI = 0.197$ in	$= 0.501$ cm
$\delta_{AISC,linear} = 0.197$ in → **value above verified**	$= 0.502$ cm → **value above verified**
#2 – Discuss member nonlinearity by shape for when member is held in a distorted position	

The bending moment, $M_{P-\delta}(x)$, is caused by eccentric axial force along the length of the column after the initial (linear) analysis iteration. $M_{P-\delta}(x)$ is a function of the axial load (constant), and the lateral deflection. The lateral deflection changes along the column length and is a cubic function of *L* and *x*.

The nonlinearity by shape of the column after the 1st analysis iteration (linear analysis) will be emulated by assuming that $M_{P-\delta}(x = L/2 = MH)$ equals the midspan moment of a sine distribution line load.



$$q = q_0 \sin\frac{\pi x}{L}$$

$$M_{midspan} = \frac{q_o L^2}{\pi^2}$$

$$\delta_{midspan} = \frac{q_0 L^4}{\pi^4 EI}$$

$$\xrightarrow{yields} \quad \delta_{P-\delta,MH} = \frac{M_{P-\delta} L^2}{\pi^2 EI}$$

Example 3 Column P-δ

#3 & #4 – 2nd-Order P-δ analysis (kips, in)

P_u = 150 k

Iteration #	2	3	4	5	6	7	
$M_{P\text{-}\delta,i} = P_u \cdot \delta_{MH}$ =	29.56	3.613	0.442	0.054	0.007	0.001	k-in
$\delta_{P\text{-}\delta,MH,i}$ =	0.0241	0.0029	0.0004	0.0000	0.0000	0.0000	in
$M_{u,new,MH,i}$ =	264.76	268.37	268.81	268.87	268.87	268.88	k-in
% increase =	12.57%	1.365%	0.165%	0.020%	0.002%	0.000%	

$M_{u,2ndorder}$ = 268.9 k-in $\delta_{2ndorder} = \delta_{MH,linear} + \Sigma\, \delta_{P\text{-}\delta,MH}$ = 0.225 in

#5 – Repeat 2nd-Order analysis for remaining 2X axial loads

The rudimentary 2nd-order analyses shown above were repeated for the 2X remaining axial loads. See below for the results of the rudimentary analyses and the published "correct" values from AISC.

		Rudimentary 2nd-order analysis (k-in, in)	AISC solution (k-in, in)	N = # of iterations for ≤ 0.50% increase
P_u = 150 k	$M_{u,MH}$	268.9	269	4
	% increase in M_u	14.3%	14.4%	
	$\delta_{2ndorder}$	0.225	0.224	
	% increase in δ	13.9%	13.7%	
P_u = 300 k	$M_{u,MH}$	313.4	313.0	5
	% increase in M_u	33.3%	33.1%	
	$\delta_{2ndorder}$	0.261	0.261	
	% increase in δ	32.4%	32.4%	
P_u = 450 k	$M_{u,MH}$	374.9	375.0	6
	% increase in M_u	59.4%	59.4%	
	$\delta_{2ndorder}$	0.311	0.311	
	% increase in δ	57.8%	57.8%	

Example 3 Column P-δ

#3 & #4 – 2nd-Order P-δ analysis (kN, mm)

P_u = 667 kN

Iteration #	2	3	4	5	6	7	
$M_{P\text{-}\delta} = P_u \cdot \delta_{MH}$ =	3.34	0.408	0.050	0.006	0.001	0.000	kN-m
$\delta_{P\text{-}\delta,MH}$ =	0.6114	0.0747	0.0091	0.0011	0.0001	0.0000	mm
$M_{u,new,MH}$ =	29.92	30.33	30.38	30.38	30.39	30.39	kN-m
% increase =	12.56%	1.363%	0.164%	0.020%	0.002%	0.000%	

$M_{u,2ndorder}$ = 30.4 kN-m $\delta_{2ndorder} = \delta_{MH,linear} + \Sigma\, \delta_{P\text{-}\delta,MH}$ = 5.70 mm

#5 – Repeat 2nd-Order analysis for remaining 2X axial loads

The rudimentary 2nd-order analyses shown above were repeated for the 2X remaining axial loads. See below for the results of the rudimentary analyses and the published "correct" values from AISC.

		Rudimentary 2nd-order analysis (kN-m, mm)	AISC solution (kN-m, mm)	N = # of iterations for ≤ 0.50% increase
P_u = 667 kN	$M_{u,MH}$	30.4	30.4	4
	% increase in M_u	14.3%	14.4%	
	$\delta_{2ndorder}$	5.70	5.71	
	% increase in δ	13.9%	14.1%	
P_u = 1334 kN	$M_{u,MH}$	35.4	35.4	5
	% increase in M_u	33.2%	33.2%	
	$\delta_{2ndorder}$	6.62	6.63	
	% increase in δ	32.3%	32.5%	
P_u = 2001 kN	$M_{u,MH}$	42.4	42.4	6
	% increase in M_u	59.3%	59.5%	
	$\delta_{2ndorder}$	7.89	7.91	
	% increase in δ	57.7	58.0%	

Example 3 Column P-δ
#6 – Check results, try approximate method, and check assumptions

AISC states that "In confirming the accuracy of the analysis method, both moments and deflections should be checked… and in all cases should agree within 3% and 5%, respectively". The following is the error of the rudimentary 2nd-order analysis as compared to the AISC-provided exact solutions:

P_u	$M_{u,2ndorder}$	$\delta_{2ndorder}$
150 k | 667 kN	0.05%	0.23%
300 k | 1334 kN	0.14%	0.07%
450 k | 2001 kN	0.03%	0.03%

The low errors tabulated above verify the rudimentary 2nd-order analysis.

Try moment magnifier approximate method:

α = ratio of applied axial compression to Euler critical buckling $= \frac{PL^2}{\pi^2 EI}$

Q = magnification factor for moment and movement $= \frac{1}{1-\alpha}$

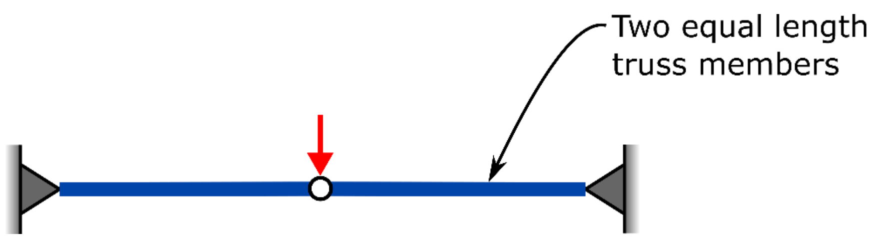
	α	Q_{approx}	$Q_{AISC,M}$	$Q_{AISC,\delta}$
150 k | 667 kN	0.122	1.139	1.144	1.137
300 k | 1334 kN	0.245	1.324	1.332	1.325
450 k | 2001 kN	0.367	1.579	1.594	1.578

The accuracy of the approximate method is remarkable. This is a very simple problem and is posed by AISC only to help benchmark SAS with 2nd-order analysis capability so that verified SAS can then be used to analyze much more complex structures. The approximate method proves to be an excellent alternative for singular members and simple structural systems, however.

Check assumptions:

1. Check small-deflection assumption: max deflection = L/1080 → **assumption valid**
2. Check material elasticity: M/S + P/A = 375/70.2 + 450/14.1 = 37.3 ksi (257 MPa), which is less than the F_Y = 50 ksi (345 MPa) for ASTM A992 steel → **material remains elastic**

Example 4 – Biot's truss	
Included Nonlinearity	Geometric; kinematics required to analyze
Possible Nonlinearity	Material nonlinearity; yielding and post-yielding response
Brief	A pin supported, 2-member truss that is *instantaneously changeable*. The middle joint has no initial resistance to vertical translation and only begins to provide translational resistance after finite translation (vertical displacement). Kinematics (or dynamics) is required to analyze this structural system.



H = span =	400 in	1,020 cm
T_{PRE} = truss member pretensioning force	1,000 lbf	4,450 N
P_u = vertical force at middle joint =	70 lbf	311 N
A (approx. 0.20" or 5 mm diameter rod) =	0.0127 in^2	8.19 mm^2
E =	10,000 ksi	69 GPa

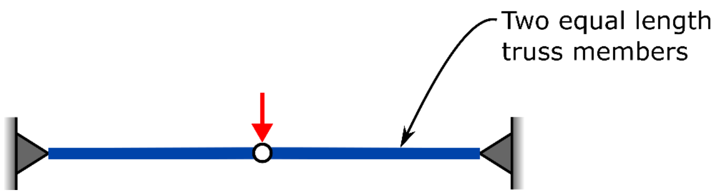
Assume no material nonlinearity, i.e. that member stresses remain elastic

Methodology

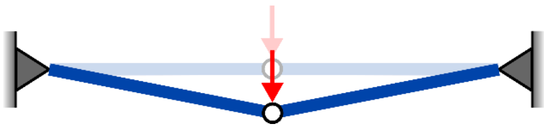
1. Discussion and formulation of truss member kinematics
2. Perform manual 2nd-order kinematic analysis
3. Compare results with benchmark solution and discuss

Example 4 Biot's truss

#1 – Discussion and formulation of truss member kinematics



This truss (above) is typically considered to be statically, internally unstable. The joint between the two truss members can infinitesimally translate vertically without any resistance from the truss structure.

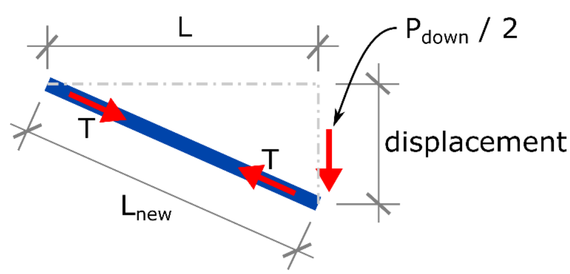


Kinematically this truss is stable. Vertical stiffness emerges once the joint translates any finite distance, though that stiffness is negligible initially. The joint would translate down vertically until a kinematic equilibrium state is established, when tensile strain in the members and internal forces are resolved. To increase stiffness the truss members can be pretensioned, e.g. each member composed of 2X threaded rods with a turnbuckle body between.

This is an example of geometric nonlinearity and a special case since the structure must displace (move) to be able to be analyzed. This is a classic problem that has a few different formats; this specific configuration is known as Biot's truss and is often used as a benchmark problem for kinematics. The problem parameters mentioned above match those used in previous solutions of the problem.

Kinematics

As the middle node translates downwards the truss members will be engaged in tension and will elastically elongate. Derive the equilibrium equations for a rotated truss member:



$L_{new} = (L^2 + displacement^2)^{1/2}$ $L_{delta} = L_{new} - L$ $Elongation = T_{delta}L/AE$

$T = (P_{down}/2)*(L_{new}/displacement)$ $T_{delta} = T - T_{pre}$

Example 4 Biot's truss

#2 – Perform manual 2nd-order kinematic analysis

Analysis Parameters Convergence = Elongation/L_{delta}
unity indicates convergance, use target precision of 1% = 0.01
the "Disp" displacement values below were manual user inputs

Manual 2nd-Order analysis

(in, lbf)	Disp	Lnew	Ldelta	T	Tdelta	Elong	Conv
	2	200.01	0.01000	3500.2	2500.17	3.93728	394
	4	200.04	0.04000	1750.3	750.35	1.18165	29.5
	6	200.0900	0.08998	1167.2	167.19	0.26329	2.93
btwn 6.0 & 8.0	8	200.1599	0.15994	875.7	-124.30	-0.19575	-1.224
	7	200.1225	0.12246	1000.6	0.61	0.00096	0.008
	6.5	200.1056	0.10560	1077.5	77.49	0.12203	1.156
btwn 6.5 & 6.6	6.6	200.1089	0.10887	1061.2	61.18	0.09635	0.885
	6.52	200.1062	0.10625	1074.2	74.19	0.11683	1.100
	6.54	200.1069	0.10690	1070.9	70.91	0.11167	1.045
Result =	**6.56**	200.1076	**0.10756**	1067.6	67.65	**0.10653**	**0.990**

Manual 2nd-Order analysis

(mm, N)	Disp	Lnew	Ldelta	T	Tdelta	Elong	Conv
	50	5080.246	0.2461	15819.9	11371.7	102.3	416
	100	5080.984	0.9842	7911.1	3462.9	31.1	31.6
	150	5082.214	2.2141	5275.3	827.14	7.4379	3.359
btwn 150 & 200	200	5083.935	3.9355	3957.8	-490.36	-4.4095	-1.120
	160	5082.519	2.5191	4945.9	497.73	4.4757	1.777
	170	5082.844	2.8437	4655.3	207.09	1.8622	0.655
btwn 165 & 170	165	5082.679	2.6789	4796.2	348.00	3.1293	1.168
	167	5082.744	2.7442	4738.8	290.62	2.6134	0.952
	166	5082.711	2.7115	4767.3	319.14	2.8698	1.058
Result =	**166.5**	5082.728	**2.7278**	4753.0	304.84	**2.7412**	**1.005**

Example 4 Biot's truss

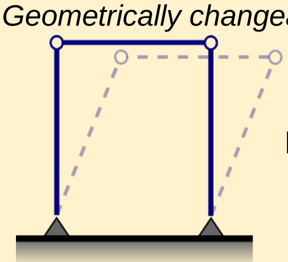
#3 – Compare results with benchmark solution and discuss

The solution[19] is:

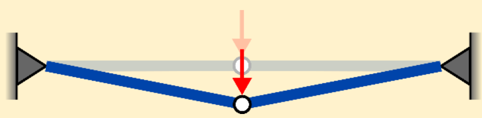
6.55654 inches = 166.53 mm **→ rudimentary 2nd-order kinematic analysis verified**

This problem was shown to illustrate nonlinear analysis and because it is a classic benchmark problem but *instantaneously changeable* structures should not be directly used in conventional building and bridge structures, and a *geometrically changeable* structural system should never be directly used in conventional structural engineering practice[20]. (Karnovsky and Lebed recommend that a structural system like Biot's truss be called *changeable* rather than *unstable*, and that the notion of stable/unstable be reserved for relating to critical load.)

Geometrically changeable refers to a structural system that can finitely distort (displace, rotate) without member deformation and is also commonly referred to as internally unstable. This condition is like the final state of a pushover analysis, while an

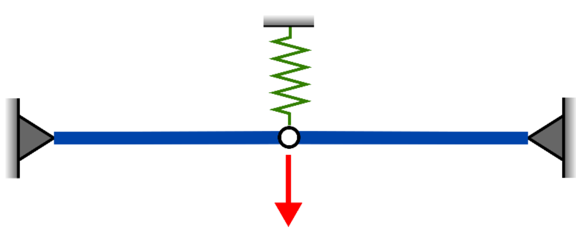


instantaneously changeable structure can infinitesimally distort without member deformation but becomes *geometrically unchangeable* and directly develops internal member forces thereafter.



However, the analyses of instantaneously changeable structural systems could be of use in the secondary analyses of structural systems. For example, Biot's truss could be encountered in the context of conventional structural engineering if it had initially been a pinned-pinned single-member beam that was loaded until a plastic hinge developed at midspan.

Biot's truss is sometimes analyzed with a vertical, translational spring at the middle joint. The stiffness rate, *k*, of the spring can be adjusted to measure linear structural analysis software's sensitivity to rigid body motion.



Most simulation (and 3rd order analysis, and multi-physics) software and analysis software with structural dynamics capability can solve instantaneously changeable structures without stabilizing springs, but software with only 2nd-order analysis capability typically cannot.

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Images

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