A SunCam online continuing education course

Combined Stress and Mohr's Circle

by

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Contents

1.0 Introduction

Stress analysis is a fundamental part of engineering. It is critical that engineers understand stresses caused by different loading conditions and determine the location of maximum stress. Fundamental mechanics of materials teaches stress caused by basic loading. The stress caused by the three basic loading types are defined in Equation 1 through Equation 3.

Axial Stress: $$\sigma_a = \frac{P}{A}$$ Equation 1

Torsional Stress: $$\tau = \frac{Tr}{J}$$ Equation 2

Flexural Stress: $$\sigma_f = \frac{My}{I}$$ Equation 3

Axial stress, as defined in Equation 1, is defined as the axial load P divided by the cross-sectional area A. Torsional stress is defined in Equation 2 for a round bar with radius r, applied torque T, and the cross-section's polar moment of inertia J. For a solid round bar, the polar moment of inertia will be

$$J = \frac{\pi r^4}{2}$$

The flexural stress (or bending stress) for a beam is defined in Equation 3, where M is the bending moment, y is the location on the cross-section away from the neutral axis, and I is the cross-section's moment of inertia. The distribution of flexural stress is shown in Figure 1 (a) for a general case. Notice that the flexural stress is zero at the neutral axis and becomes larger at the extreme fibers (y becomes larger). The bending moment M is determined using moment diagrams, and its value will depend on the loading and the location along the beam length. Figure 1 (b) illustrates a shear V and moment M diagram for a basic simply supported beam with a single point load. Figure 1 (c) shows the sign convention for bending moment.

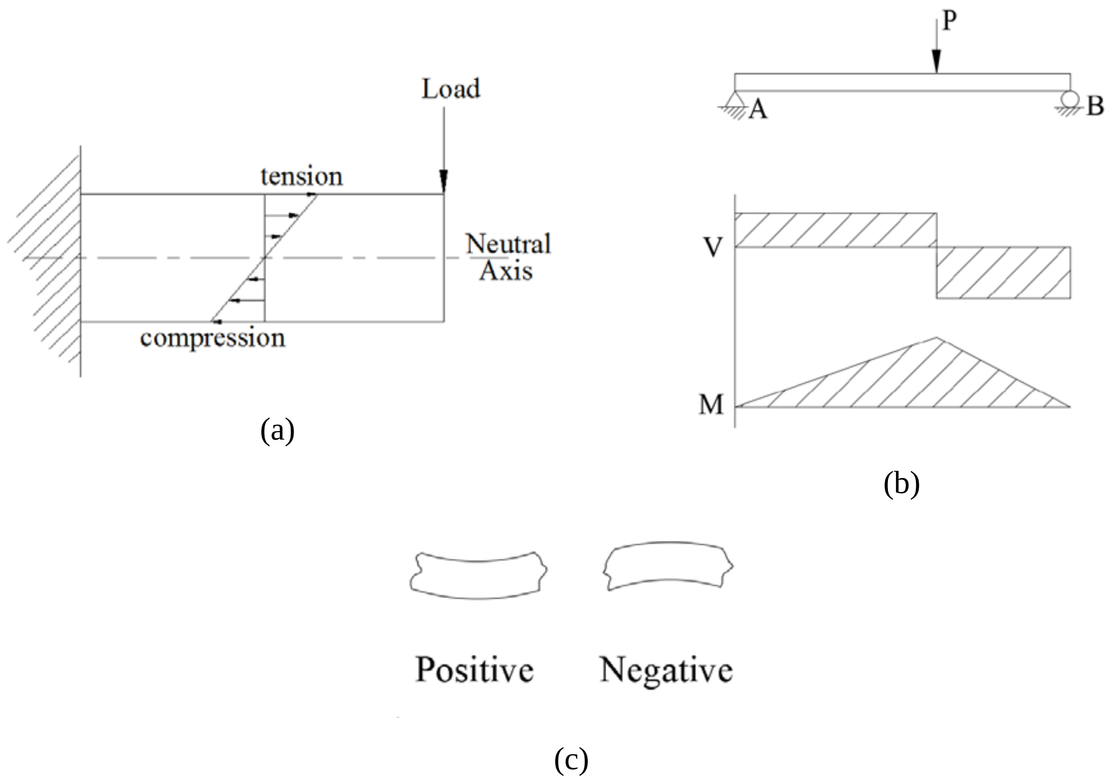


Figure 1 (a) General flexural stress distribution (b) Example of shear and moment diagram (c) Sign convention for bending moment

This course will expand on the basic loading types to explore combinations of basic loading types. Combined stress analysis is based on a concept of taking a cubic unit from any point. The forces acting on this cube are determined based on the loading, and the cube will be in equilibrium. The member will contain multiple cubes, so the analysis process includes finding stresses at a critical point. These concepts will be illustrated throughout this course for different types of combined loading.

2.0 Combination of Axial and Flexural Loads

We will begin with a basic, yet common, example of a combination of axial and flexural loading. Consider the simply supported beam shown in Figure 2 (a), which has a concentrated load causing flexural stress. For this case the fibers above the neutral axis will be in compression (-), the fibers below the neutral axis will be in tension (+), and the stress at the neutral axis will be zero. The stress distribution is shown. The same beam is shown in Figure 2 (b) with an axial tension load. The axial load will cause the uniform stress shown, and the entire cross-section will have a tension stress (+).

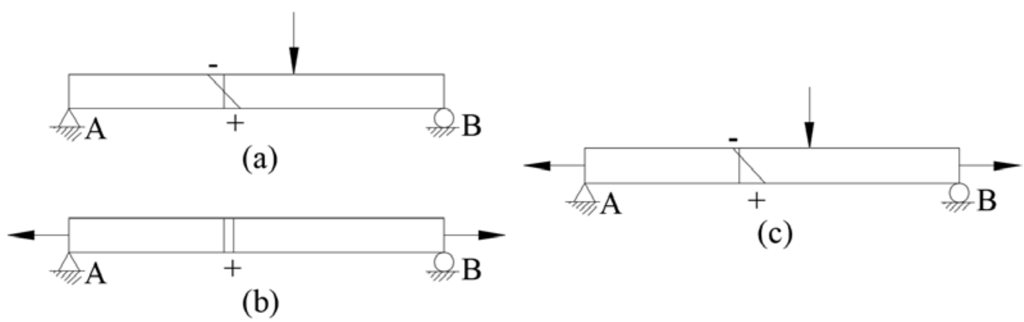


Figure 2 (a) Flexural stress (b) Axial stress (c) Combined flexural and axial stress

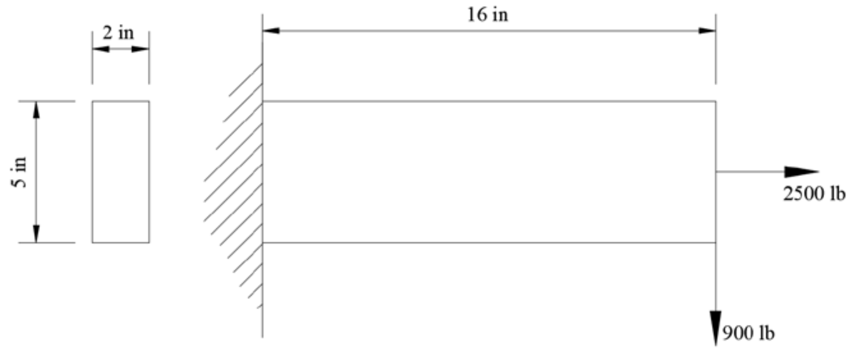
Figure 2 (c) shows the effects of combining the bending and axial loading. The resulting stress will be the superposition of the two separate stresses. Because all stresses act normal to the cross-section, the combined stress is simply the axial stress plus the flexural stress.

$$\sigma = \sigma_a + \sigma_f \qquad \text{Equation 4}$$

Notice that the stresses all act normal to the cross-section, but they may be compressive or tensile. Therefore, pay close attention to signs.

Example 1

A cantilever beam with a rectangular cross-section bar carries the loading shown. Determine the maximum stress.



Solution:

This is an example of a combination of flexural stress and axial tensile stress. The axial stress will be determined using Equation 1.

$$\sigma_a = \frac{P}{A} = \frac{2500lb}{2in(5in)} = 250psi$$

The axial stress is considered uniform, and will be the same value over the entire length of the beam. The bending stress will be maximum at the fixed end, because that is the location of the maximum bending moment. The moment at the fixed end will be

$$M = 900lb(16in) = 14{,}400in \cdot lb$$

The cross-section's moment of inertia will be

$$I = \frac{bh^3}{12} = \frac{2in(5in)^3}{12} = 20.83in^4$$

Therefore, the bending stress is determined using Equation 3.

$$\sigma_f = \frac{My}{I} = \frac{14{,}400in \cdot lb(2.5in)}{(20.83in)^4} = 1{,}728psi$$

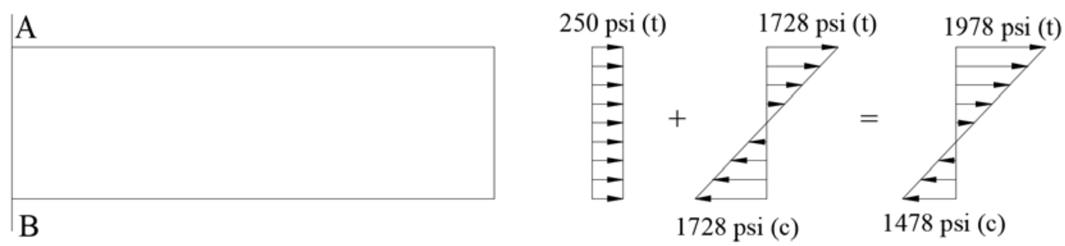
Because of the direction of bending, the upper fibers will be in tension and the lower fibers will be in compression. The combined stress values are determined from Equation 4. For point A, the combined stress will result in a tensile stress of

$$\sigma_A = 250psi + 1728psi = 1{,}978psi$$

For point B, the combined stress will result in a compressive stress of

$$\sigma_B = 250psi - 1728psi = 1{,}478psi$$

The stress distribution for line A-B is shown below, where (t) indicates tension and (c) indicates compression.



The neutral axis (location of zero stress) for the standard bending stress will be located at the cross-section's centroid. However, for the combined loading case it is not located at the centroidal axis of the cross-section. For this example, it shifts down by a value of *y*, which can be determined by calculating the location of the combined stress equal to zero.

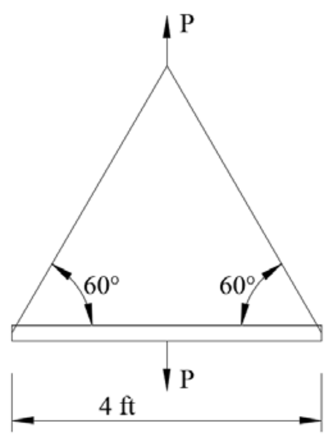
$$0 = \frac{P}{A} - \frac{My}{I} = 250\frac{lb}{in^2} - \frac{(14400in \cdot lb)y}{(20.83in)^4}$$

$$y = 0.36in$$

Therefore, the neutral axis moves down 0.36 inches from the centroid.

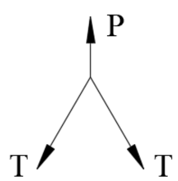
Example 2

A lifting beam is constructed from 6-inch square tube (area equals 5.59 in^2 and moment of inertia equals 10.1 in^4). For the configuration shown, determine the maximum load *P* that can be lifted if the stress in the tube is not to exceed 20 ksi.



Solution:

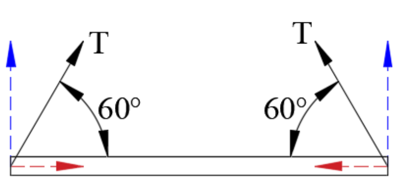
The axial force in the beam is computed in terms of *P* using the tension in the cables. The cable tension is determined from the free body diagram shown. Summation of vertical forces gives



$$P - 2T \sin 60 = 0$$

$$T = \frac{P}{2 \sin 60} = 0.5774P$$

Applying the tension force to the ends of the beam will give the axial force in the beam, which will be a compressive force.



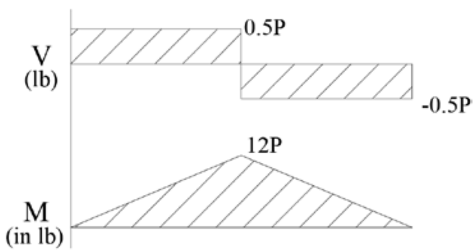
The red arrows indicate the axial force, which is the horizontal component of the tension.

$$T_x = T \cos 60$$

Substituting the original expression for T would give the axial force in terms of the applied load P. The axial force acting on each end of the beam is

$$F_a = 0.5774P\cos 60 = 0.2887P$$

The vertical component of the tension will equal 0.5P, which is the beam reaction load. The maximum bending moment will occur under the concentrated load.



From the shear (V) and moment (M) diagram, the maximum moment will be the area under the shear diagram.

$$M = 0.5P(24in)$$

$$M = 12P$$

The maximum combined stress will be compressive because the axial load is in compression. Therefore, the maximum stress will occur on the top surface of the beam located at the midpoint (location of maximum bending moment). From Equation 4, using the maximum allowed stress

$$-\sigma_{allowable} = \frac{F_a}{A} - \frac{My}{I}$$

$$-20000\frac{lb}{in^2} = \frac{-0.2887P}{5.59in^2} - \frac{(12P)(3in)}{10.1in^4}$$

Solving for P gives

$$P = 5{,}531lb$$

In the previous example, the beam was subjected to bending and an axial compression load. In the case of combining bending with axial compression, the axial load can cause additional deflection (the additional deflection was assumed negligible in the previous example). For short compression members, members having a low slenderness ratio, the additional deflection caused by compression can be ignored. If the member becomes long, the effects of the additional deflection become more significant.

Consider the beam shown in Figure 3. The concentrated load, shown in blue, causes bending in the beam and a deflected shape shown in blue. Adding the axial compression load, shown in red, causes additional deflection resulting the in the red deflected shape.

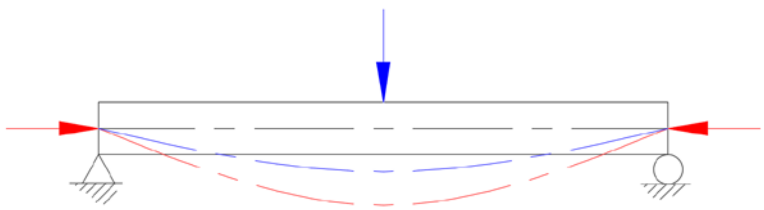


Figure 3 Effects of compression load on bending

Another case exists that causes a constant bending moment. Consider the beam shown in Figure 4 (a). The axial loads *P* are applied off center by a distance *e*. The eccentric loading will cause a bending moment, as shown in Figure 4 (b). The constant bending moment will equal the load multiplied by the eccentricity.

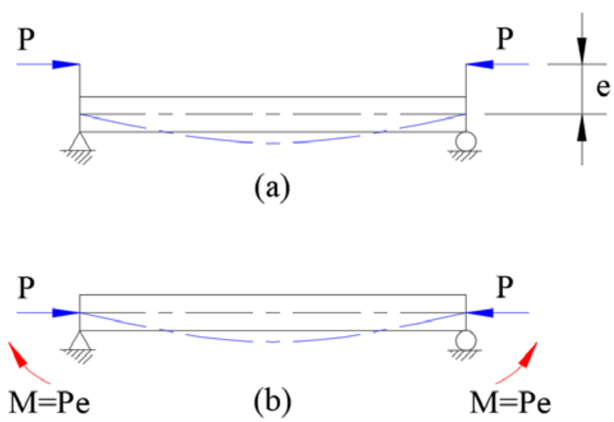


Figure 4 (a) Eccentric axial compression load (b) Resulting combined axial compression and constant moment

For either the case in Figure 3 or Figure 4, an amplification factor can be determined using Figure 5. A beam can have any transverse loading, but the *k* factor can be determined using the plots based on the type of axial compressive load.

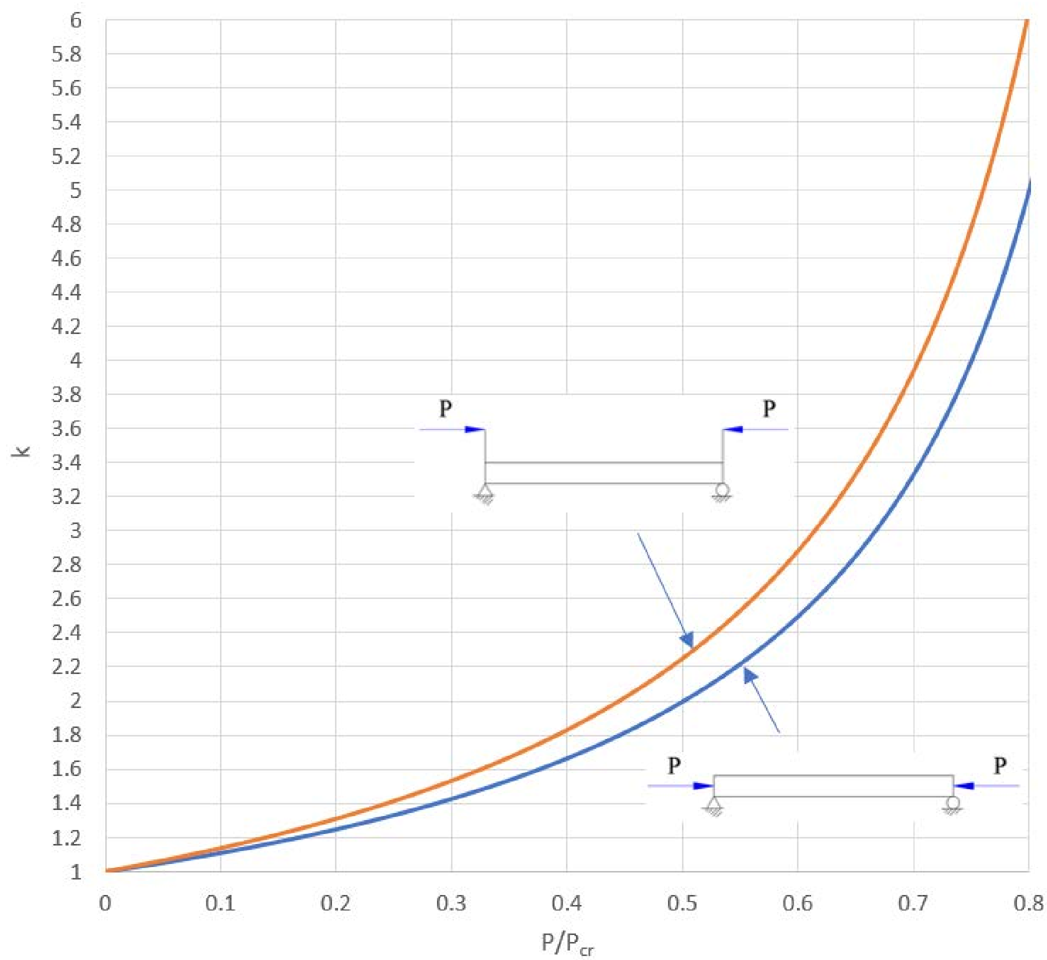


Figure 5 Amplification factor

The horizontal axis of Figure 5 is a term for P/P_{cr}, where P is the applied load and P_{cr} is the critical buckling load determined from Equation 5.

Critical Buckling Load $$P_{cr} = \frac{EI\pi^2}{L^2}$$ Equation 5

In Equation 5 E is the modulus of elasticity, I is the moment of inertia, and L is the beam length.

After the amplification factor is determined, an interaction curve can be used to determine if the combined loading is safe. The interaction curve shown in Figure 6 uses two ratios comparing the actual loading to the ultimate loading.

$$R_a = \frac{P}{P_u}$$ Equation 6

$$R_b = \frac{kM}{M_u}$$ Equation 7

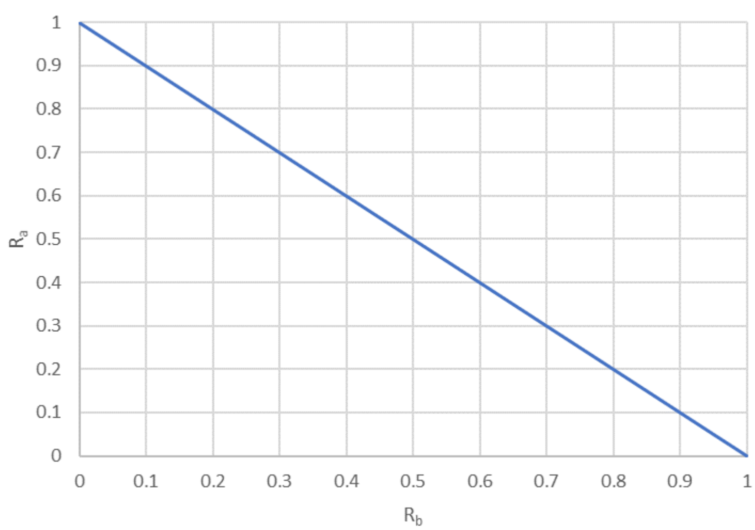


Figure 6 Interaction curve for combined axial compression and bending

For a specific loading condition, the ratios can be calculated and plotted on the interaction curve. Any point plotted below the blue curve indicates a safe loading condition.

The ratios require the ultimate loads. For Equation 7, the ultimate moment is the plastic moment. The plastic moment is when all fibers in the cross-section have yielded. Consider the rectangular beam cross-section shown in Figure 7. The stress distribution is shown once all fibers have reached the yield stress F_y. A moment couple is created from the forces shown, which gives a plastic moment (or ultimate moment) of

$$M_u = \left(bF_y \frac{d}{2}\right)\frac{d}{2} = F_y \frac{bd^2}{4} \qquad \text{Equation 8}$$

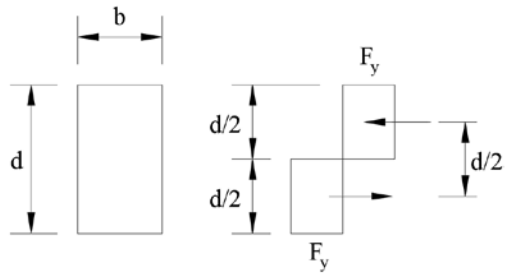
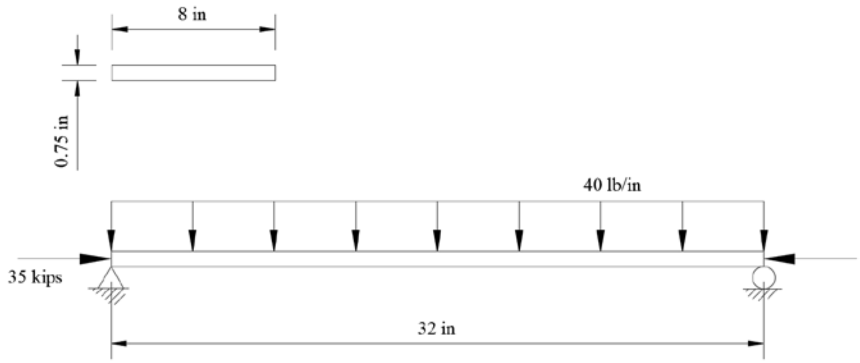


Figure 7 Plastic moment

The ultimate axial load in Equation 6 is based on column equations and depends on the column's slenderness ratio (length divided by radius of gyration). Depending on the slenderness ratio, the column falls into one of three categories: short column, intermediate column, or long column. The critical buckling load from Equation 5 is based on long columns, so the analysis in this course will focus on long columns. Long columns have a slenderness ratio above 100. Therefore, for $\frac{L}{r} < 100$, the critical load equation is not valid. Complete coverage of columns is beyond the scope of this course, so the example will be based on long column conditions where the ultimate axial load is assumed to be the critical buckling load. Note that the member from Example 2 would have been a short column (L/r = 35.7), so the effects of additional bending from compression are ignored.

Example 3

A machine element has a width of 8 inches and a thickness of 0.75 inches. The 32-inch long element carries a uniform load of 40 lb/in and has an axial compression load of 35 kips located at the centroid. The modulus of elasticity is 30x10^6 psi and the yield strength is 36 ksi. Determine if the combined loading is safe.



Solution:

The cross-sectional properties are

$$A = bh = 8in(0.75in) = 6in^2$$

$$I = \frac{bh^3}{12} = \frac{8in(0.75in)^3}{12} = 0.281in^4$$

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.281in^4}{6in^2}} = 0.2165in$$

Beginning with the axial loading, we need to know the critical load from Equation 5.

$$P_{cr} = \frac{EI\pi^2}{L^2} = \frac{\left(30000000\frac{lb}{in^2}\right)(0.281in^4)\pi^2}{(32in)^2} = 81{,}323lb$$

The amplification factor is determined using the ratio of the applied load to the critical load.

$$\frac{P}{P_{cr}} = \frac{35{,}000 lb}{81{,}323 lb} = 0.43$$

Using the blue curve from Figure 5 gives

$$k = 1.80$$

Next, we calculate the bending moment due to the uniform load. Shear and moment diagrams can be used, or the equation can be determined from beam tables. For a uniform load distribution, the maximum moment will occur at the beam's midpoint and have a value of

$$M = \frac{wL^2}{8} = \frac{\left(40\, {}^{lb}/_{in}\right)(32in)^2}{8} = 5{,}120 in \cdot lb$$

The actual moment caused by the extra deflection due to axial compression is determined by

$$kM = 1.80(5{,}120 in \cdot lb) = 9{,}216 in \cdot lb$$

The combined stress due to bending and axial compression is determined by the superposition of the stresses.

$$\sigma = \frac{P}{A} \pm \frac{kMy}{I}$$

$$\sigma = \frac{35{,}000 lb}{6in^2} \pm \frac{9{,}216 in \cdot lb (0.375 in)}{0.281 in^4}$$

$$\sigma = 5{,}833.3 psi \pm 12{,}288 psi$$

Therefore, the maximum compressive stress is 18,121 psi and the maximum tensile stress is 6,455 psi. The interaction curve is used to determine if the combined loading is safe. The ratios from Equation 6 and Equation 7 must be used, but the ultimate loads are not known. For the ultimate axial load, the member has a high slenderness ratio (L/r is approximately 148) so the critical load can be used as the ultimate load.

$$R_a = \frac{P}{P_u} = \frac{35{,}000lb}{81{,}323lb} = 0.43$$

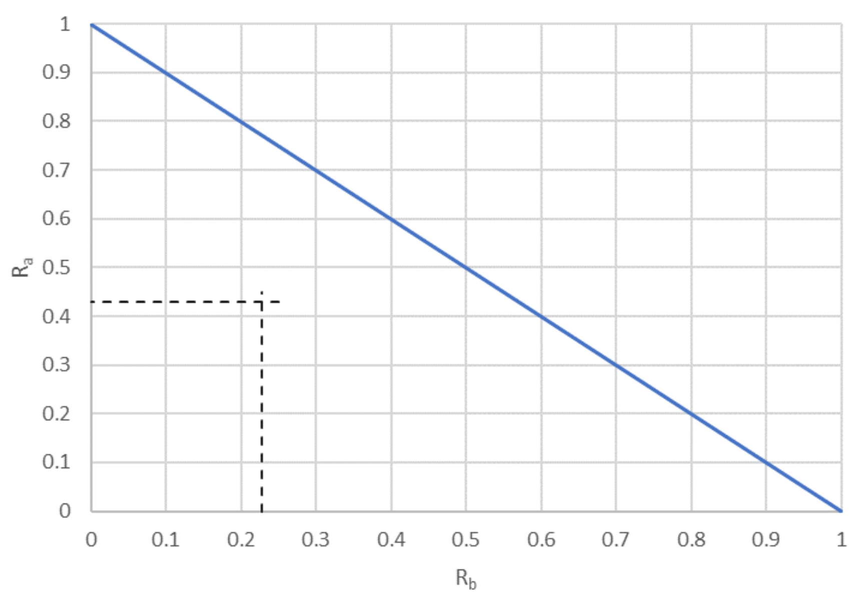
The ultimate moment is based on the plastic bending moment. Using Equation 8

$$M_u = F_y \frac{bd^2}{4} = 36{,}000 \frac{lb}{in^2} \left(\frac{8in(0.75in^2)}{4} \right) = 40{,}500 in \cdot lb$$

Using Equation 7

$$R_b = \frac{kM}{M_u} = \frac{9{,}216 in \cdot lb}{40{,}500 in \cdot lb} = 0.23$$

The ratios can then be plotted on the interaction curve shown below. The intersection of the ratios falls below the blue curve indicating that the member is safe.



3.0 Stress at a Point

3.1 Introduction

The previous section was a very specific example of combined stress caused by bending and axial loading (which allowed for superposition methods of analysis), but stress combinations can happen in many forms. We will see that stress at a point for general loading will combine normal stress and shear stress. Normal stress, as shown in Figure 8, acts normal (at right angles) to the reference plane and has the symbol σ. The direction shown is tension and an inward direction would be compression. Shear stress, which is also shown in Figure 8, acts parallel to the reference plane and has the symbol τ.

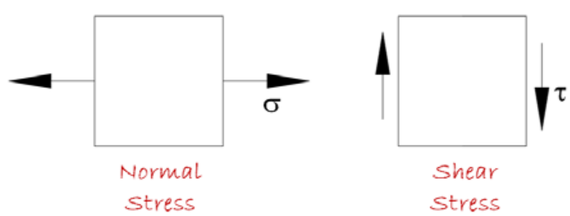


Figure 8 Normal stress and shear stress

Stress at a point can be defined by components acting in several directions, as shown in Figure 9. Normal forces will have a single subscript indicating the face on which it acts. For example, the *x* subscript indicates a normal force acting on the *x* face of the element. Shear forces have a double subscript, as shown in Figure 9. The first subscript indicates the face on which it acts and the second indicates the direction. For example, the subscript *xy* would indicate a shear force acting on the *x* face and in the *y*-axis direction.

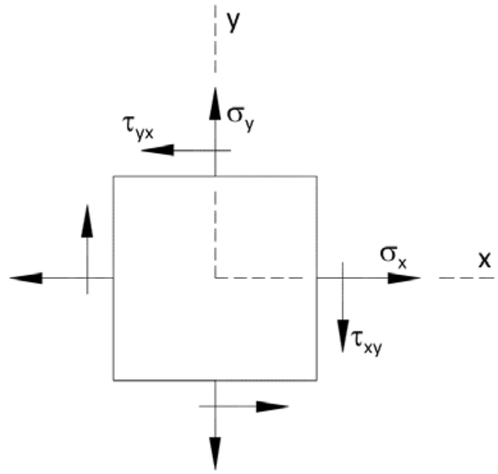


Figure 9 Stress components

3.2 Stress on an Oblique Plane

Consider an axially loaded bar, as shown in Figure 10 (a). If a section is cut perpendicular to the axial direction, the normal stress will be as shown in Figure 10 (b). However, a section can be cut on an oblique plane as shown in Figure 10 (c). The resultant force shown in Figure 10 (d) would then have two components: a normal component N and a shear component V.

$$N = P\cos\theta \quad \text{Equation 9}$$

$$V = P\sin\theta \quad \text{Equation 10}$$

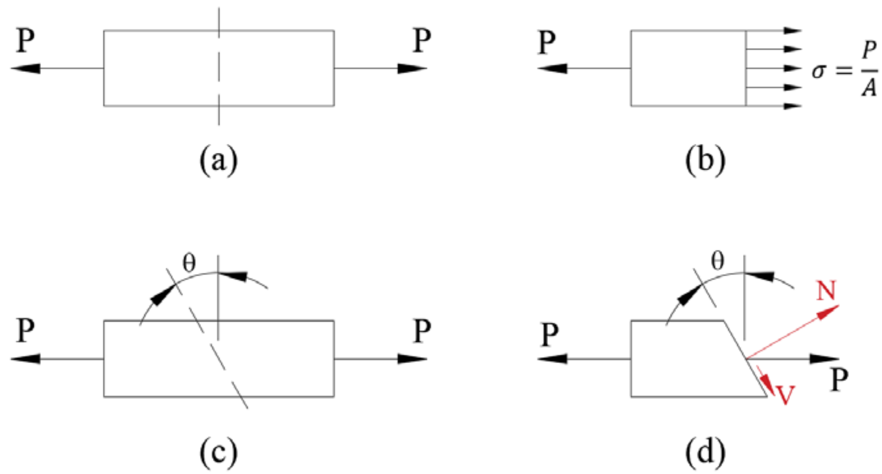


Figure 10 (a) Axially loaded bar with normal plane (b) Stress on normal plane (c) Axially loaded bar with oblique plane (d) Component forces on oblique plane

3.2.1 Normal Stress

Looking first at the normal force on the oblique plane in Figure 10 (d), we can develop an equation for normal stress as a function of θ. Consider a bar with a width *w* and a height *h* as shown in Figure 11. For an oblique plane, such as the one shown in Figure 11 (b), the length of the side will be *h*/cosθ. Therefore, the shaded area for the oblique plane would be

$$A_n = w\frac{h}{\cos\theta}$$

Noting that the product of *w* and *h* is equal to the normal plane area, as shown in Figure 11 (a), the normal area can be written as

Normal Area $$A_n = \frac{A}{\cos\theta}$$ Equation 11

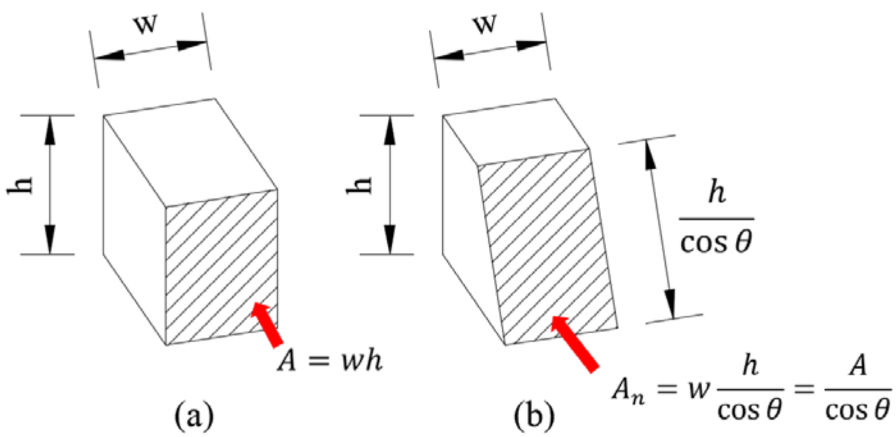


Figure 11 (a) Normal plane (b) Oblique plane

The normal stress will be the normal force divided by the normal area. Using Equation 9 and Equation 11, the normal stress is given by

$$\sigma_n = \frac{N}{A_n} = P\cos\theta\left(\frac{\cos\theta}{A}\right)$$

Normal Stress $$\sigma_n = \frac{P}{A}\cos^2\theta$$ Equation 12

The normal stress will be maximum when θ is zero degrees.

3.2.2 Shear Stress

Similarly, the shear stress can be determined by the shearing force divided by the normal area. Using Equation 10 and Equation 11 gives

$$\tau = \frac{V}{A_n} = P\sin\theta\left(\frac{\cos\theta}{A}\right)$$

$$\sigma_n = \frac{P}{A}\sin\theta\cos\theta$$

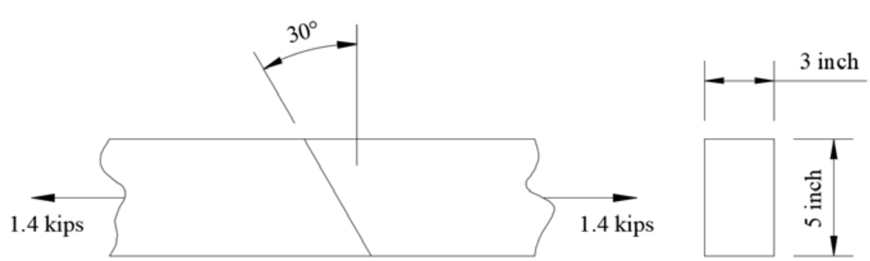
Using the trig identity $\sin 2\theta = 2\sin\theta\cos\theta$ gives

Shear Stress $$\tau = \frac{P}{2A}\sin 2\theta$$ Equation 13

The shear stress will be maximum when θ is forty-five degrees.

Example 4

A wooden bar has a cross-section of 3 inches wide and 5 inches tall. A glued joint occurs at a 30-degree angle as shown. What is the normal stress and shear stress on the joint for a 1.4 kip axial load?



Solution:

The normal stress is calculated using Equation 12.

$$\sigma_n = \frac{P}{A}\cos^2\theta = \frac{1400lb}{3in(5in)}\cos^2 30$$

$$\sigma_n = 70psi$$

The shear stress is calculated using Equation 13.

$$\tau = \frac{P}{2A}\sin 2\theta = \frac{1400lb}{2(3in)(5in)}\sin 60$$

$$\tau = 40.4psi$$

3.3 Variation of Stress at a Point

Previous sections focused on rather specific example cases of combined stresses. General equations exist that allow solutions to all possible combinations of plane stresses. The general equations are developed by taking a general stress at a point, such as that illustrated in Figure 9, and passing a plane through the element cutting it into two pieces. The cutting plane is at an angle θ. From equations of equilibrium (without proof), the general equations for normal and shear stress become

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad \text{Equation 14}$$

$$\tau_n = \tau_{xy}\cos 2\theta - \frac{\sigma_x - \sigma_y}{2}\sin 2\theta \qquad \text{Equation 15}$$

To illustrate the equations, consider a point shown in Figure 12 (a) with $\sigma_x = 25ksi$, $\sigma_y = 7ksi$, and $\tau_{xy} = 12ksi$. Using these values in Equation 14 and Equation 15 gives the plots shown in Figure 12 (b).

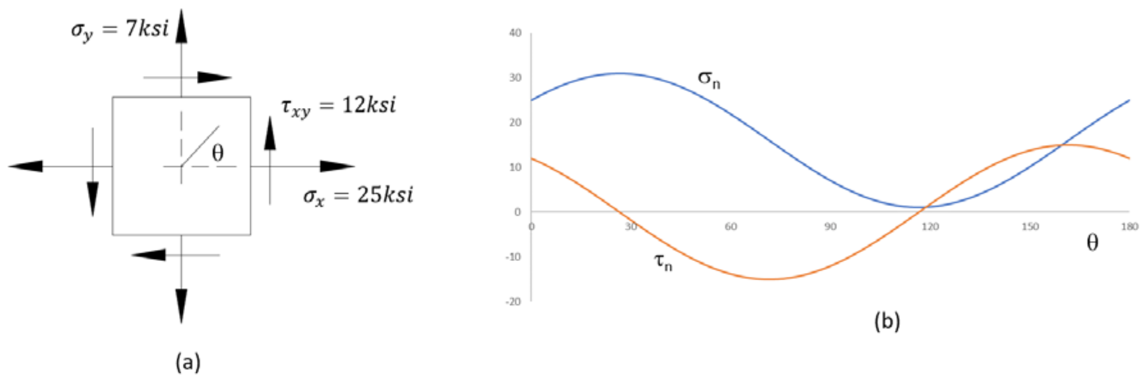


Figure 12 (a) Example stress at a point (b) Variation of stress

3.4 Principal Stress

The maximum and minimum stresses at a point are known as the principal stresses. Referring to the previous plot in Figure 12 (b) as an example, the maximum and minimum normal stress occurs when the shear stress is equal to zero. Setting Equation 15 equal to zero gives

$$\tau_n = 0 = 12\cos 2\theta - \frac{25-7}{2}\sin 2\theta$$

which gives the maximum normal stress at $\theta = 26.56°$ and minimum at $\theta = 116.56°$. Notice these points coincide with zero shear stress on the plot, as well as the locations of maximum and minimum normal stress.

Maximum and minimum normal stresses (principal stresses) occur on planes of zero shearing stress. Therefore, the planes on which the principal stresses act are called principal planes.

3.5 Mohr's Circle

Mohr's circle is a graphical tool to illustrate how stresses combine, and it is a tool used for finding the principal stresses discussed in Section 3.4. For a general stress element, such as that shown in Figure 9, Mohr's circle can be created using the following steps:

<u>Steps for Developing Mohr's Circle</u> – Reference Figure 13:

1. Start with a rectangular σ-τ coordinate system
2. Plot the points (σ_x,τ_{xy}) and (σ_y,τ_{yx})
    a. For normal stress – tension is positive and compression is negative
    b. For shear stress – shearing stress is positive when its moment about the element center is clockwise
3. Join the plotted points with a straight line. The line gives the center of the circle and the radius, which can be calculated using

Center $$C = \frac{\sigma_x + \sigma_y}{2}$$ Equation 16

Radius $$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$ Equation 17

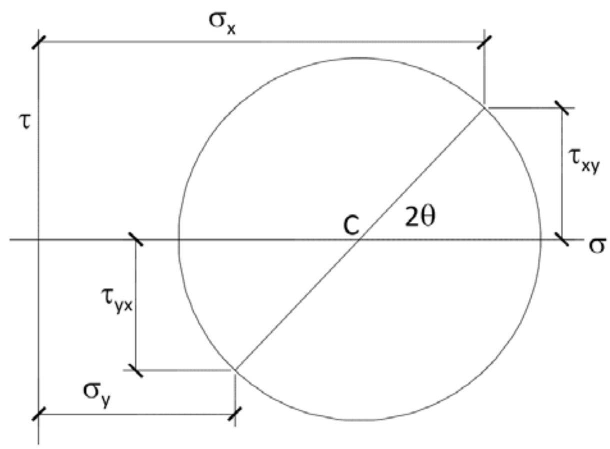


Figure 13 Mohr's circle

It can be seen from Figure 13 that the maximum normal stress will occur at the center plus the radius of the circle. Similarly, the minimum normal stress will occur at the center minus the radius. Using Equation 16 for the center and Equation 17 for the radius, the maximum and minimum values are

$$(\sigma_n)_{max} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{Equation 18}$$

$$(\sigma_n)_{min} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{Equation 19}$$

The maximum and minimum normal stresses are known as the principal stresses σ_1 and σ_2. Any angle of rotation on Mohr's circle will cause the planes to rotate a value of half the angle in the same direction. For example, Figure 14 shows that the angle between σ_1 and σ_2 will be 180° on Mohr's circle but only 90° on the stress element. Therefore, notice that the angle shown in Figure 13 is defined as 2θ.

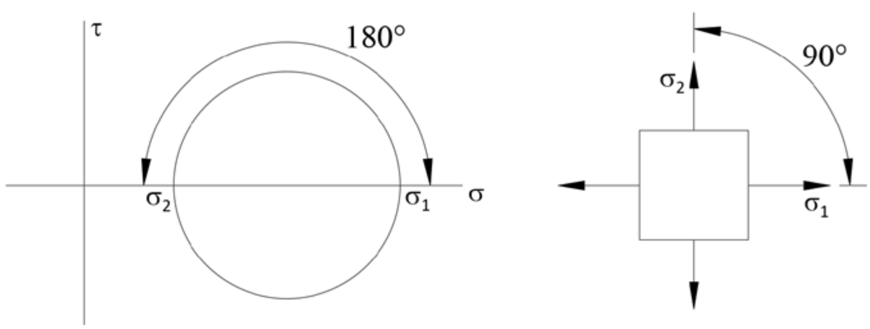


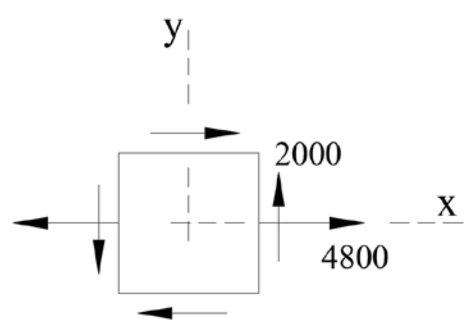
Figure 14

The maximum shear stress will occur at the top point of the circle, which is equal to the radius of Mohr's circle.

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{Equation 20}$$

Example 5

The stress element shown has $\sigma_x = 4800psi$, $\sigma_y = 0$, and $\tau_{xy} = -2000psi$. Determine the principal stresses, maximum shear stress, and the planes on which these stresses act.



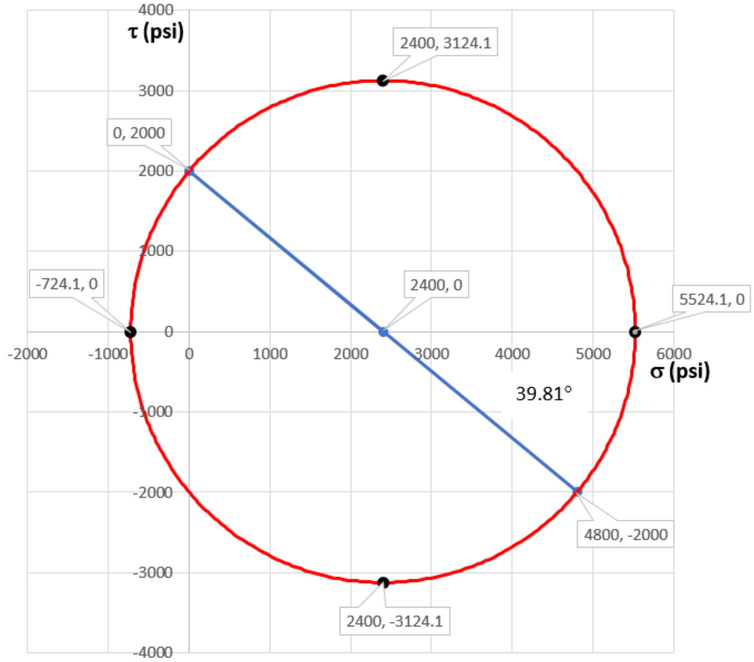
Solution:

Development of Mohr's circle begins with plotting the points

$$(\sigma_x, \tau_{xy}) = (4800, -2000)$$

$$(\sigma_y, \tau_{yx}) = (0{,}2000)$$

Connecting the points gives the blue line in the figure below, which is the completed Mohr's circle.



The center, Equation 16, will be located along the horizontal axis at the location

$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{4800 + 0}{2} = 2400$$

The circle, shown in red, can then be drawn through the two points with the center defined. The radius of Mohr's circle is defined using Equation 17 to give

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{4800 - 0}{2}\right)^2 + (-2000)^2} = 3124.1$$

Equation 18 and Equation 19 give the principal stresses.

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 2400 + 3124.1$$

$$\sigma_1 = 5524.1 psi$$

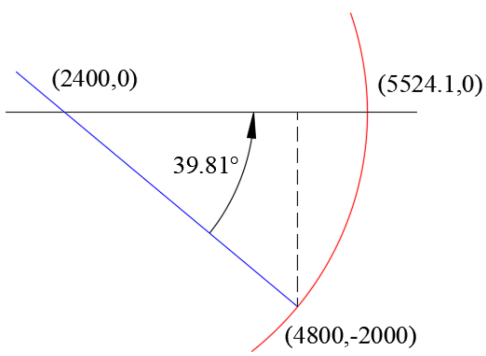
$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 2400 - 3124.1$$

$$\sigma_2 = -724.1psi$$

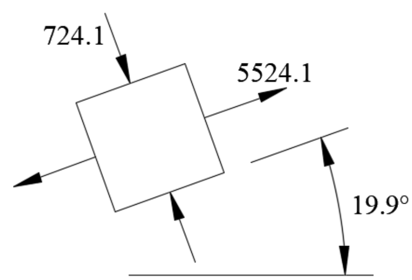
Both principal stresses are shown on Mohr's circle where the red circle crosses the horizontal axis. For the principal plane, we can refer to the portion of Mohr's circle shown in figure (a) below. The angle can be calculated as

$$\tan 2\theta = \frac{2000}{5524.1 - 2400}$$

which gives a value of 2θ of 39.81°. The element will rotate half of that angle to give the principal planes shown in figure (b) below.



(a)

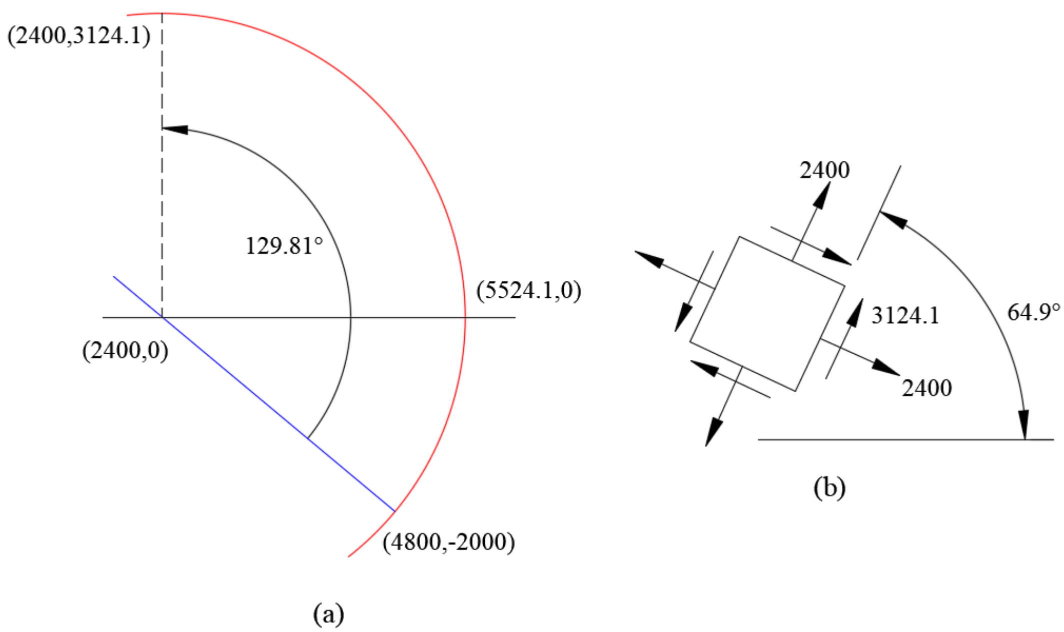


(b)

The maximum shear stress occurs at the top of Mohr's circle, which is equal to the radius of the circle.

$$\tau_{max} = 3124.1psi$$

The angle to the maximum shear stress is shown in figure (a) below and will equal the previous principal plane axis of 39.81° plus 90° to give 129.81°. The plane for maximum shear stress will be half of the angle to give the figure shown in (b) below.



3.6 Absolute Maximum Shearing Stress

Equations for Mohr's circle can determine a maximum in-plane shear stress, but it only considers one flat plane. There are two other planes that are not considered in the equations. Consider the element shown in Figure 15. For convenience, the coordinate axes are aligned with the principal stresses.

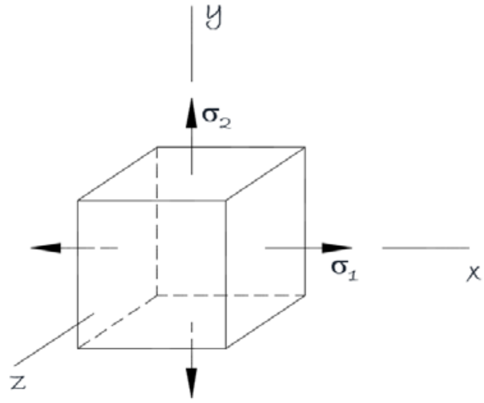


Figure 15

Looking at the x-y plane gives the Mohr's circle shown in Figure 16.

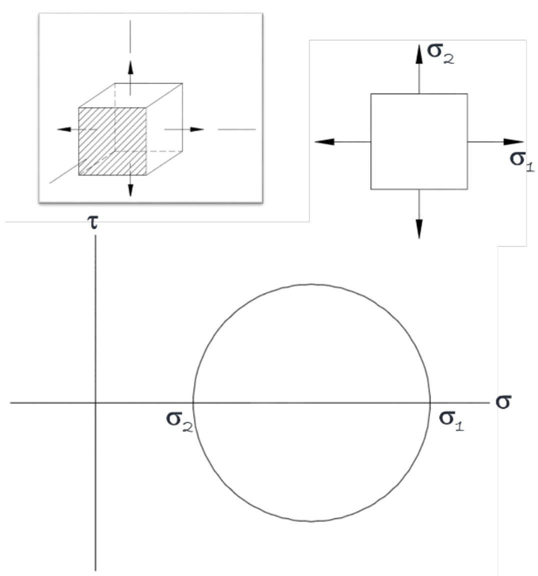


Figure 16 Mohr's circle for x-y plane

Looking at the x-z plane gives the Mohr's circle shown in Figure 17.

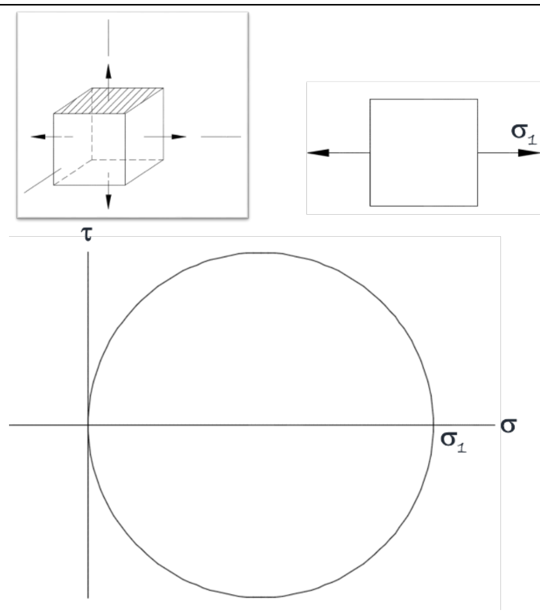


Figure 17 Mohr's circle for x-z plane

Looking at the y-z plane gives the Mohr's circle shown in Figure 18.

Combining all planes in one figure gives the Mohr's three-circle diagram shown in Figure 19. The figure is also known as a 3D Mohr's circle. It can be determined from the figure that, for this example, the maximum shear stress comes from the x-z plane.

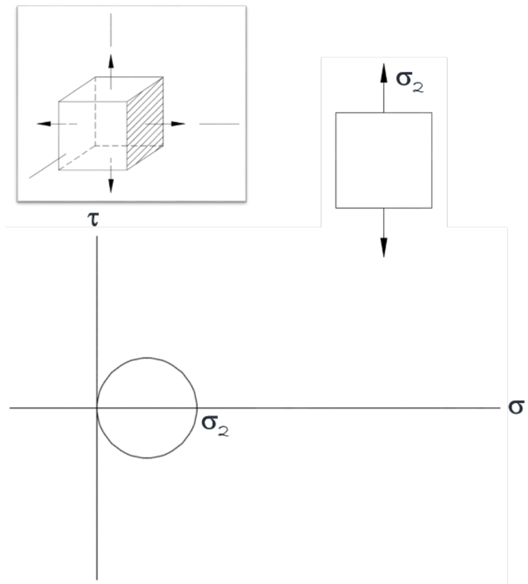


Figure 18 Mohr's circle for y-z plane

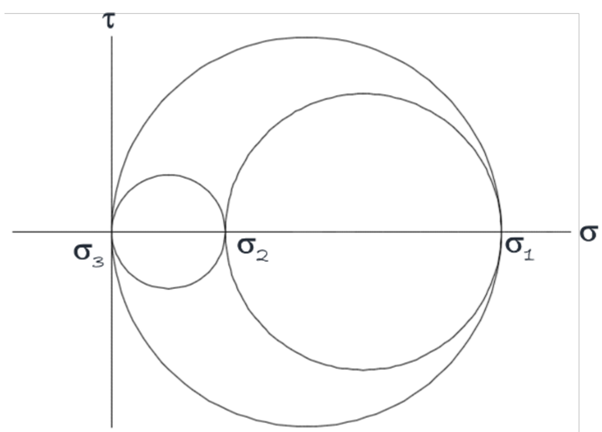


Figure 19 Combined Mohr's circle for all planes

The absolute maximum shear stress will equal the radius of the largest circle. Therefore, the maximum shearing stress will equal the largest of the following values:

$$\frac{|\sigma_1 - \sigma_2|}{2}$$
$$\frac{|\sigma_2 - \sigma_3|}{2} \qquad \text{Equation 21}$$
$$\frac{|\sigma_3 - \sigma_1|}{2}$$

4.0 Applications of Mohr's Circle

4.1 Combined Axial Load and Torsion

A common loading condition combines axial loading and torsional loading. The combined stress will include axial and torsional stress given by Equation 1 and Equation 2, respectively. Consider, as an illustrative example, a 3-inch diameter shaft with an axial loading of 35 kips and a torsional loading of 1000 ft-lb. The shaft will have an area equal to

$$A = \frac{\pi d^2}{4} = \frac{\pi (3in)^2}{4} = 7.07in^2$$

and a polar moment of inertia equal to

$$J = \frac{\pi d^4}{32} = \frac{\pi (3in)^4}{32} = 7.95in^4$$

First consider the axial loading alone, which is shown in Figure 20 (a).

$$\sigma_a = \frac{P}{A} = \frac{35000lb}{7.07in^2} = 4951.5psi$$

The stress element will only have a tensile load in the *x*-axis direction as shown.

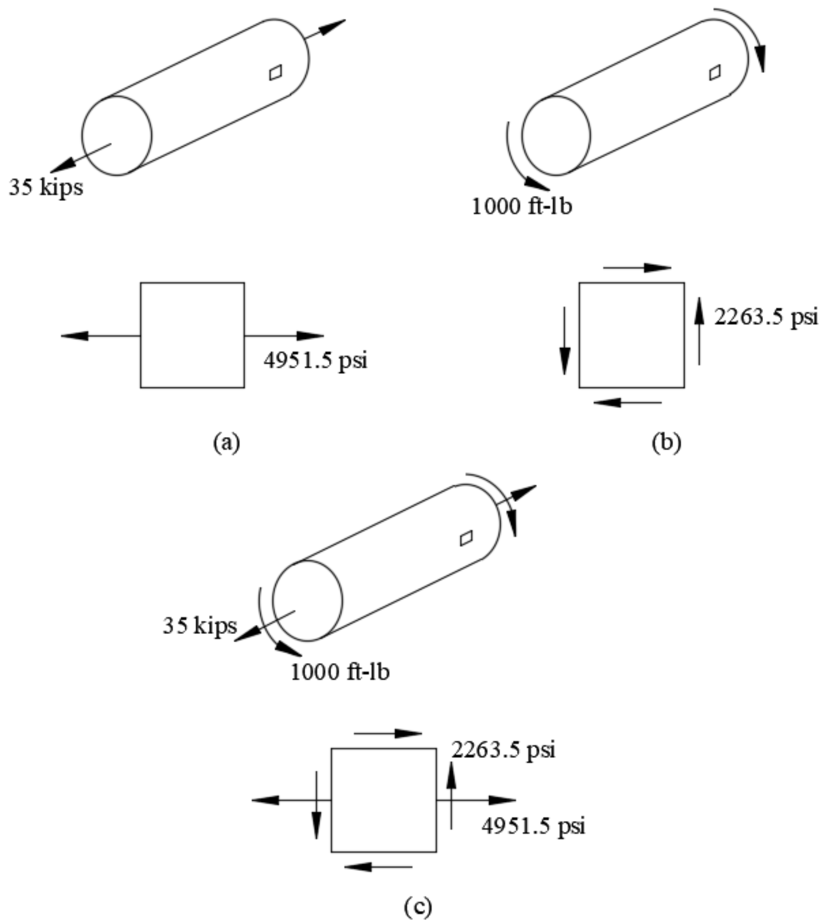


Figure 20 Stress elements for loadings (a) Axial loading only (b) Torsional loading only (c) Combined loading

The stress element for axial loading only will result in the Mohr's circle shown in Figure 21.

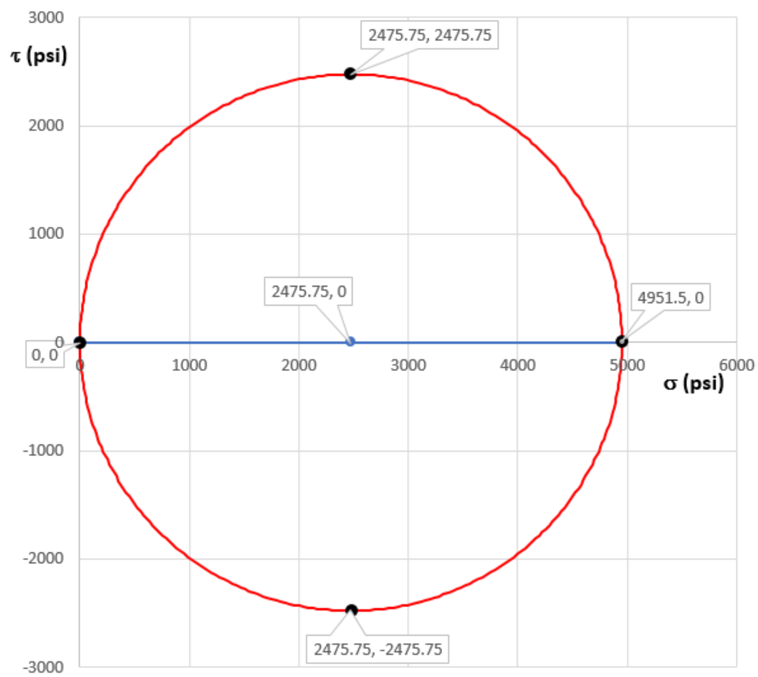


Figure 21 Mohr's circle for axial loading only

Next, consider the torsional loading alone as shown in Figure 20 (b). The torsional stress will be

$$\tau = \frac{Tr}{J} = \frac{1000ft \cdot lb\left(12\frac{in}{ft}\right)(1.5in)}{7.95in^4} = 2263.5psi$$

The torsional stress will give the stress element shown in Figure 20 (b) and will have the Mohr's circle shown in Figure 22.

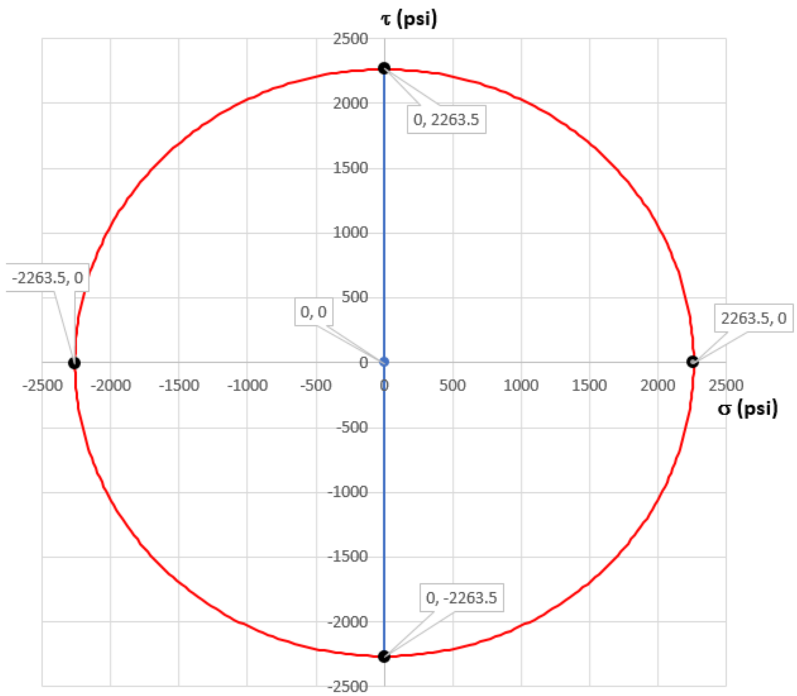


Figure 22 Mohr's circle for torsional loading only

Combining the loading will give the stress element shown in Figure 20 (c). The Mohr's circle will have a center at

$$C = \frac{\sigma_x + \sigma_y}{2} = \frac{4951.5 + 0}{2} = 2475.75$$

and will have a radius equal to

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{4951.5 - 0}{2}\right)^2 + 2263.5^2} = 3354.51$$

The resulting Mohr's circle is shown in Figure 23.

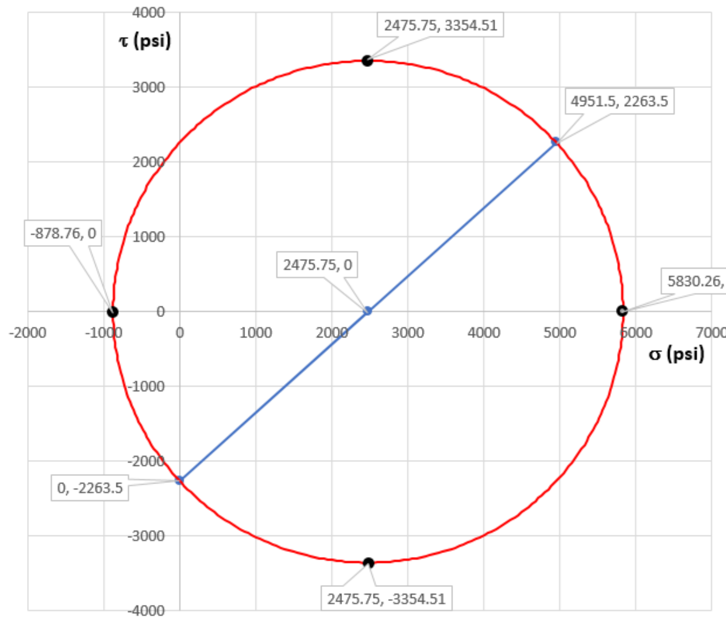


Figure 23 Mohr's circle for combined axial and torsional loading

4.2 Combined Bending and Torsion

Another common application of Mohr's circle is a combination of bending and torsion. Shafts, for example, are commonly subjected to torsional and bending loads. The combined stress will include torsional and flexural stresses given in Equation 2 and Equation 3, respectively. The combination will be developed similarly to the case for combined axial and torsional loading because bending will also cause axial loads. A typical stress element is shown in Figure 24 along with the resulting Mohr's circle. The stresses on the element are

$$\sigma_x = \frac{My}{I}$$

$$\tau_{xy} = \frac{Tr}{J}$$

The points plotted to generate Mohr's circle are

$$\left(0, -\frac{Tr}{J}\right)$$

$$\left(\frac{My}{I}, \frac{Tr}{J}\right)$$

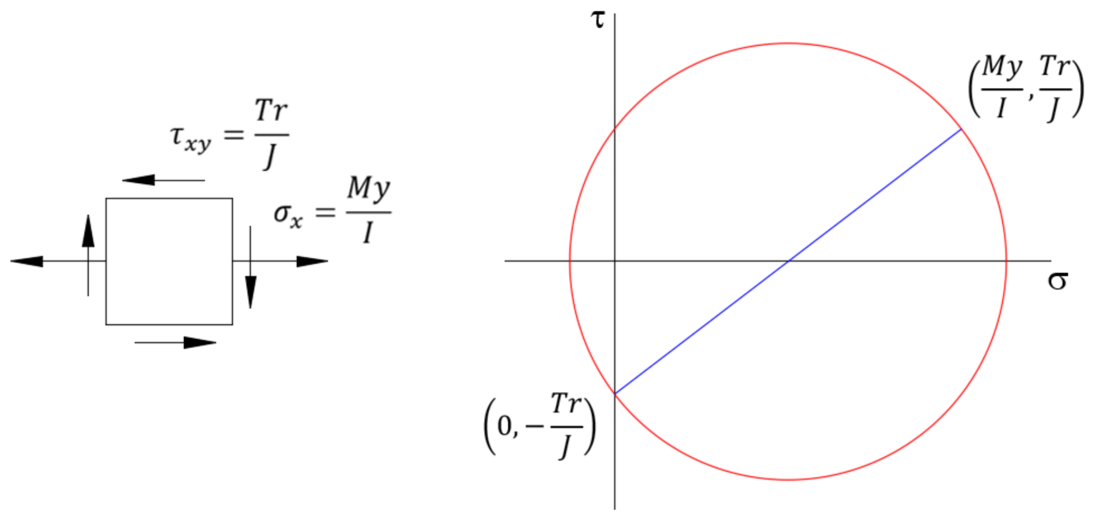


Figure 24 Typical stress element and Mohr's circle for combined bending and torsion

The radius of Mohr's circle will be

$$R = \sqrt{\left(\frac{My}{2I}\right)^2 + \left(\frac{Tr}{J}\right)^2}$$

and the center will be located at

$$C = \frac{1}{2}\left(\frac{My}{I}\right)$$

Therefore, the principal stresses will equal

$$\sigma_{1,2} = \frac{My}{2I} \pm \sqrt{\left(\frac{My}{2I}\right)^2 + \left(\frac{Tr}{J}\right)^2} \qquad \text{Equation 22}$$

and the maximum shearing stress will be equal to the radius.

$$\tau_{max} = \sqrt{\left(\frac{My}{2I}\right)^2 + \left(\frac{Tr}{J}\right)^2} \qquad \text{Equation 23}$$

General design formulas can be developed from Equation 22 and Equation 23. If we assume shear stress will govern the design, we can modify Equation 23 by using the relationship between polar moment of inertia. Due to symmetry, the moment of inertia about the *x*-axis will equal that about the *y*-axis giving

$$J = I_x + I_y = 2I$$

Substituting into Equation 23, and noting that y and r are the same value, gives

$$\tau_{max} = \sqrt{\left(\frac{Mr}{J}\right)^2 + \left(\frac{Tr}{J}\right)^2}$$

$$\tau_{max} = \frac{r}{J}\sqrt{M^2 + T^2} \qquad \text{Equation 24}$$

Using the equations for the radius and polar moment of inertia in terms of diameter gives

$$\tau_{max} = \frac{D/2}{\pi D^4/32}\sqrt{M^2 + T^2}$$

$$D^3 = \frac{16\sqrt{M^2 + T^2}}{\pi(\tau_{max})} \qquad \text{Equation 25}$$

Another design equation can be developed assuming normal stress governs the design. Modifying the maximum normal stress from Equation 22 gives

$$\sigma_1 = \frac{Mr}{2I} + \sqrt{\left(\frac{Mr}{2I}\right)^2 + \left(\frac{Tr}{2I}\right)^2}$$

$$\sigma_1 = \frac{Mr}{2I} + \frac{r}{2I}\sqrt{M^2 + T^2}$$

$$\sigma_1 = \frac{M(D/2)}{2\pi D^4/64} + \frac{D/2}{2\pi D^4/64}\sqrt{M^2 + T^2}$$

$$D^3 = \frac{16}{\pi(\sigma_1)}\left(M + \sqrt{M^2 + T^2}\right) \qquad \text{Equation 26}$$

Note that the design equations given in Equation 25 and Equation 26 are only basic design equations and should be used for static combinations of bending and torsion. Full dynamic shaft design often requires fatigue analysis and includes stress concentrations. Full coverage of shaft design is beyond the scope of this course.

Example 6

A round shaft is subjected to a bending moment of 5500 ft·lb and a torsion of 3000 ft·lb. Design the shaft if the allowable normal stress is 12 ksi and the allowable shear stress is 9 ksi.

Solution:

Equation 25 will be used first to investigate the case for allowable shear stress.

$$D^3 = \frac{16\sqrt{(66000in \cdot lb)^2 + (36000in \cdot lb)^2}}{\pi\left(9000\frac{lb}{in^2}\right)}$$

$$D = 3.5in$$

Next, Equation 26 will be used to investigate the case for allowable normal stress.

$$D^3 = \frac{16}{\pi\left(12000\frac{lb}{in^2}\right)}\left(66000in \cdot lb + \sqrt{(66000in \cdot lb)^2 + (36000in \cdot lb)^2}\right)$$

$$D = 3.9in$$

Because the required diameter for normal stress is larger, it will govern the design.

$$D_{min} = 3.9in$$