A SunCam online continuing education course

Pressure Vessels

Thin and Thick-Walled Stress Analysis

by

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Contents

1.0 Course Overview

This course covers the basic concepts of the analysis of vessels holding a fluid or gas under pressure. Some common examples of pressurized vessels include pipes, water towers, hydraulic cylinders, and boilers. This course will focus on cylindrical and spherical shaped vessels because they are common in industrial applications and straightforward to analyze.

Stress calculations in this course are separated into two categories. The first section will cover basic stresses in thin-walled pressure vessels. The second section covers stresses in thick-walled pressure vessels. The majority of the course covers topics for thick-walled pressure vessels due to the increased complexity.

Note that this course only covers stress caused by pressure loads. Hangers or support brackets can be used to support tanks or pressure vessels. The supports will cause additional stress, which when combined with the working stresses of the vessel cannot exceed an allowable limit for the material. It is also important that the support still allows flexibility of the vessel and not cause it to become too rigid. Though analytical methods exist for determining additional stresses caused by supports, it is recommended to use FEA [not covered in this course] to analyze support areas.

This course covers the basic stress analysis of pressure vessels and does not cover specific design codes for pressure vessels due to the vast types of applications. The reader should consult any appropriate codes, such as ASME Code Section VIII, for more details.

2.0 Thin-Walled Pressure Vessels

2.1 Introduction

Pressure vessels will fall into one of two main categories: thin-walled pressure vessels and thick-walled pressure vessels. A thin-walled pressure vessel has a small plate thickness compared to the overall diameter of the vessel. Consider a pressurized cylindrical vessel with an inner radius r and a wall thickness t. For cylinders having a wall thickness less than or equal to 10% of the inner radius $[t \le r/10]$, the pressure vessel is considered 'thin-walled'. It will be seen that hoop stress is assumed to be uniformly distributed over the wall thickness in thin-walled pressure vessels (with minimal error), where the distribution is non-uniform in thick-walled pressure vessels.

2.2 Stresses in Cylindrical Containers

We will begin with cylindrical shaped vessels with thin walls subjected to internal pressure. The internal pressure will cause stresses in the cylinder walls. For thin-walled pressure vessels, a point on the vessel (sufficiently far from the ends) will have hoop stress and longitudinal stress as illustrated with the free-body diagram shown in Figure 1.

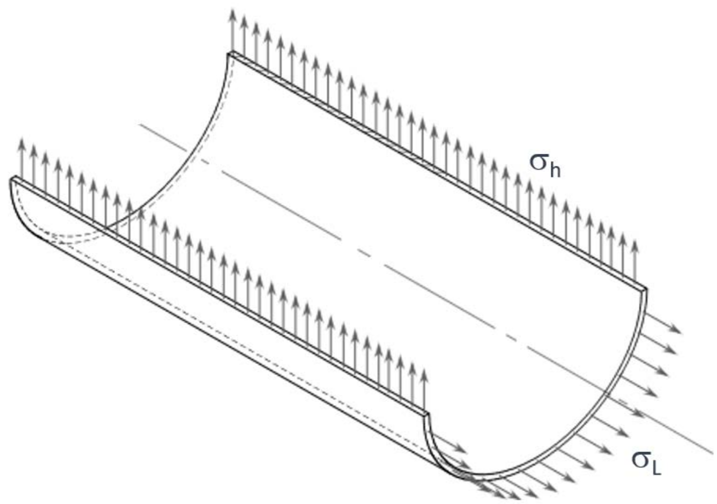


Figure 1 Illustration of hoop stress σ_h and longitudinal stress σ_L

2.2.1 Hoop Stress

For an internal pressure in the vessel we can calculate the longitudinal and hoop stress. We will begin with a derivation of hoop stress, which is a circumferential stress. Consider the cylindrical vessel shown in Figure 2 (a). The vessel contains a fluid under a pressure p [psi]. A section with a length L is cut from the cylinder and is shown isolated in Figure 2 (b).

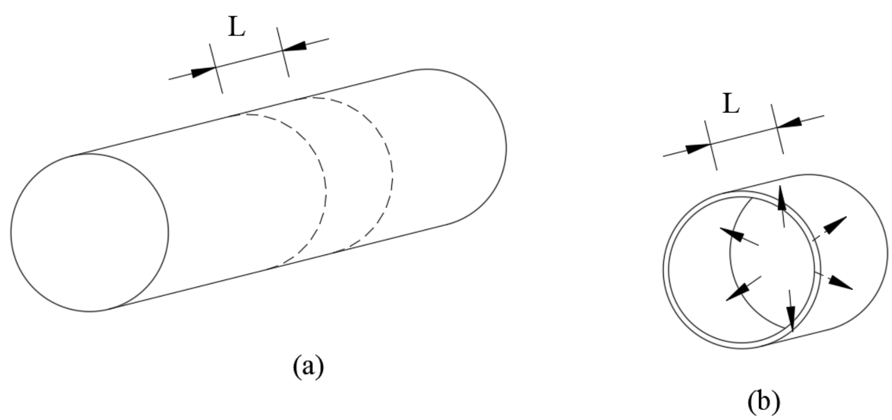


Figure 2 (a) Thin-walled pressure vessel (b) Isolated section

The section shown in Figure 2 (b) is now cut in half to determine internal forces. The half section is shown in Figure 3 (a) and has a length L, an inner diameter D, and the wall thickness is t. The internal pressure must be converted into a force, which is represented in Figure 3 (b). The resultant fluid force is equal to the fluid pressure multiplied by the area. For this case, the pressure is p and the area is the projected area of the diameter times the length.

$$F = pDL \qquad \text{Equation 1}$$

The internal force in the walls can be determined from statics. Referring to Figure 3 (b)

$$F = 2P$$

$$pDL = 2P$$

$$P = \frac{pDL}{2} \qquad \text{Equation 2}$$

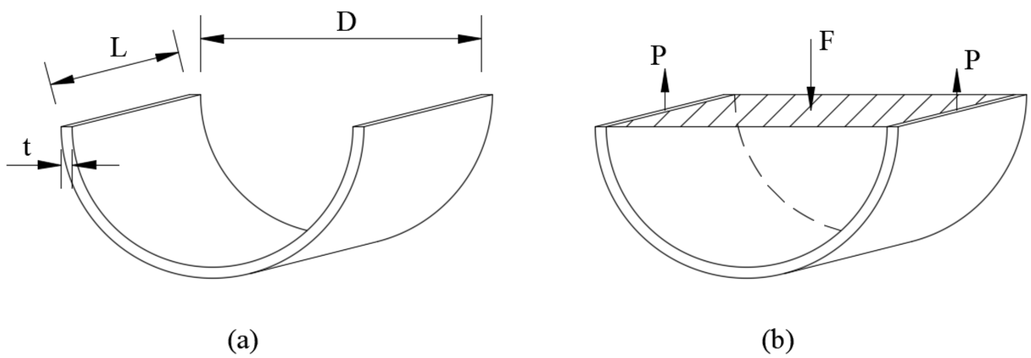


Figure 3 (a) Half section dimension (b) Forces on half section

The internal force in the walls given in Equation 2 can be used to determine the normal stress in the walls. Knowing that the stress will equal the force in the wall divided by the area of the wall, the stress is

$$\sigma_h = \frac{P}{A} = \frac{pDL}{2}\left(\frac{1}{tL}\right)$$

Hoop Stress $$\sigma_h = \frac{pD}{2t}$$ Equation 3

The hoop stress σ_h given in Equation 3 acts in the vessel's circumferential direction. In the equation p is the internal pressure in psi, D is the internal diameter of the cylinder in inches, and t is the wall thickness in inches. The resulting hoop stress will have units of psi.

Example 1

A cylindrical tank has an internal diameter of 18 inches and a wall thickness of ¼ inch. Determine the hoop stress if the internal pressure is 150 psi.

Solution:

The hoop stress is determined from Equation 3.

$$\sigma_h = \frac{pD}{2t} = \frac{150\frac{lb}{in^2}(18in)}{2(0.25in)}$$

$$\sigma_h = 5400\frac{lb}{in^2}$$

2.2.2 Longitudinal Stress

Consider the cylindrical vessel with closed ends shown in Figure 4 (a). The internal pressure will tend to push the ends of the cylinder outward causing a longitudinal tensile force in the walls. Figure 4 (b) shows a section of the cylinder with the pressure force and resulting wall forces.

The resultant force from the fluid pressure will equal the pressure times the projected area. For an internal diameter D, the force will equal

$$F = p\frac{\pi D^2}{4} \qquad \text{Equation 4}$$

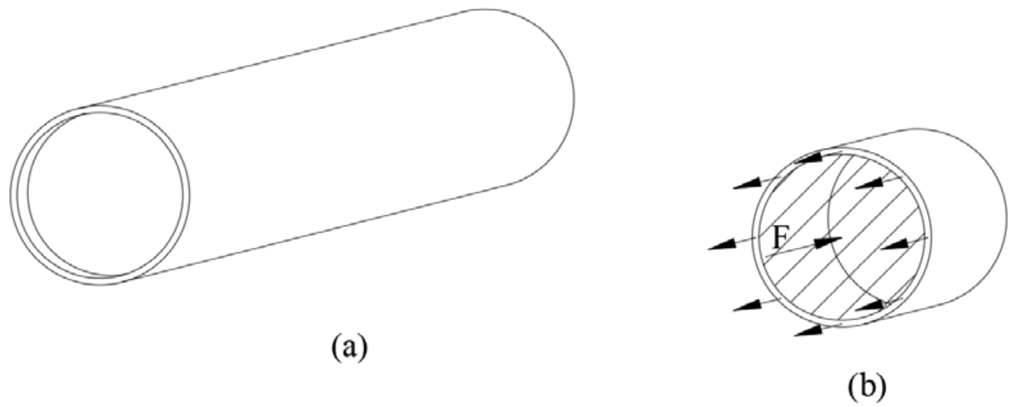


Figure 4 (a) Thin-walled pressure vessel with closed ends (b) Isolated section

The force in the cylinder wall must equal the force in Equation 4, and the internal stress will equal the force divided by the area. For a wall thickness t, the area will equal πDt.

$$\sigma_L = p\frac{\pi D^2}{4}\left(\frac{1}{\pi Dt}\right)$$

Longitudinal Stress $$\sigma_L = \frac{pD}{4t}$$ Equation 5

The longitudinal stress acts in the axial direction. In the equation p is the internal pressure in psi, D is the internal diameter of the cylinder in inches, and t is the wall thickness in inches. The resulting longitudinal stress will have units of psi.

Example 2

A cylindrical tank has an internal diameter of 26 inches and a wall thickness of ¼ inch. Determine the longitudinal stress if the internal pressure is 200 psi.

Solution:

The longitudinal stress is determined from Equation 5.

$$\sigma_L = \frac{pD}{4t} = \frac{200\frac{lb}{in^2}(26in)}{4(0.25in)}$$

$$\sigma_L = 5200\frac{lb}{in^2}$$

2.3 Stresses in Spherical Containers

Deriving stresses in the walls of a spherical container is very similar to cylindrical containers. Consider the half section of the spherical container shown in Figure 5. The container will have an internal pressure p in psi. For this case, the resultant force is the pressure times the area, which is a circular area. For a sphere inside diameter D, the force will be

$$F = p\frac{\pi D^2}{4} \qquad \text{Equation 6}$$

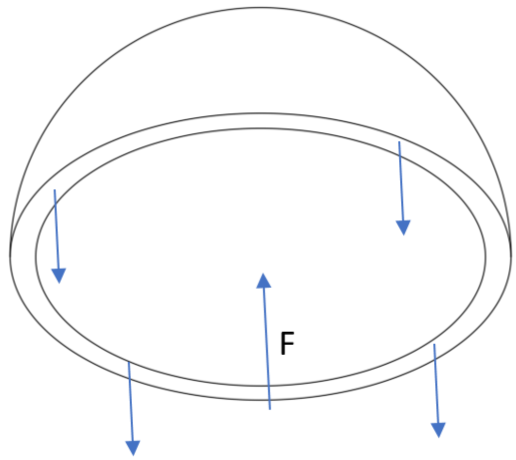


Figure 5 Half section of a spherical container

The force in the walls must equal the force from the fluid, and the internal stress will equal the force divided by the area. For a wall thickness t, the area will equal πDt.

$$\sigma = p\frac{\pi D^2}{4}\left(\frac{1}{\pi Dt}\right)$$

$$\sigma = \frac{pD}{4t} \qquad \text{Equation 7}$$

Therefore, the equation for stress in spherical vessel walls is the same as longitudinal stress in cylindrical vessels.

Example 3

A spherical pressure vessel is constructed from ½" thick plate and has an inner diameter of 3 feet. What is the maximum internal pressure if the stress cannot exceed 20 ksi? What would be the maximum internal pressure for a similarly sized cylindrical vessel?

Solution:

The maximum pressure for the spherical container can be determined using Equation 7.

$$\sigma = \frac{pD}{4t} \quad \Rightarrow \quad p = \frac{4t\sigma}{D} = \frac{4(0.5in)\left(20000\frac{lb}{in^2}\right)}{36in}$$

$$p = 1{,}111\frac{lb}{in^2}$$

For a similarly sized cylindrical container, the maximum stress occurs in the circumferential direction. From Equation 3

$$\sigma_h = \frac{pD}{2t} \quad \Rightarrow \quad p = \frac{2t\sigma_h}{D} = \frac{2(0.5in)\left(20000\frac{lb}{in^2}\right)}{36in}$$

$$p = 556\frac{lb}{in^2}$$

Therefore, a spherical pressure vessel will carry twice the internal pressure.

3.0 Thick-Walled Pressure Vessels

3.1 Introduction

The equations used to determine stress in thin-walled pressure vessels are based on the assumption that the stress distribution is uniform throughout the thickness of the cylinder walls. In reality, the stress varies over the thickness. If the walls are thin, the variation in stress is small so the stress can be assumed uniform with minimal error. For thicker walls, the variation in stress becomes more important and cannot be neglected.

This section on thick-walled vessels will be separated based on the type of loading. The first section will include pressures on the inside and outside combined, which is the most general case. The following sections will simplify equations by only considering one pressure, either internal or external.

Consider a pressure vessel with an outside radius b and an inside radius a as shown in Figure 6 (a). For the derivations of equations for stress and strain, we will use cylindrical coordinates. From Figure 6 (a) we have the axial coordinate z aligned with the central axis of the cylinder. Figure 6 (b) shows the radial and circumferential coordinates r and θ.

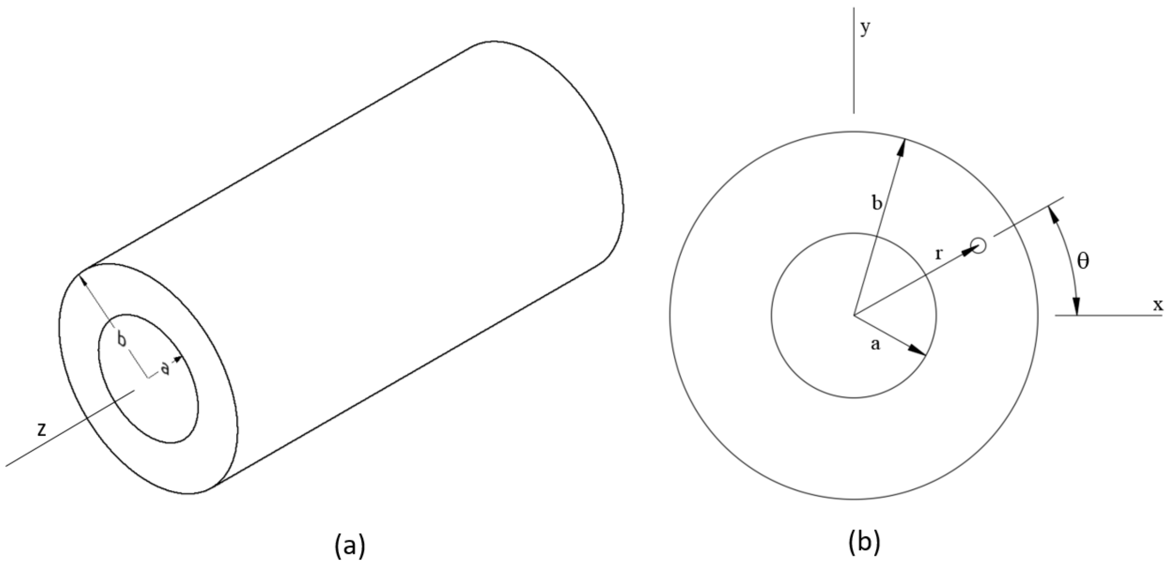


Figure 6 Thick walled pressure vessel and cylindrical coordinate system

3.2 Internal and External Pressure

We will begin with the most general case where the vessel is subjected to both internal and external pressure. The inside pressure is p_i and an outside pressure p_o as shown in Figure 7. The thick-walled cylinder can be treated as several layers of thin-walled vessels, one is shown as the dashed lines in Figure 7. One segment of a thin walled shell can be isolated, as shown in Figure 8. The figure shows a typical segment with a radius r and thickness dr.

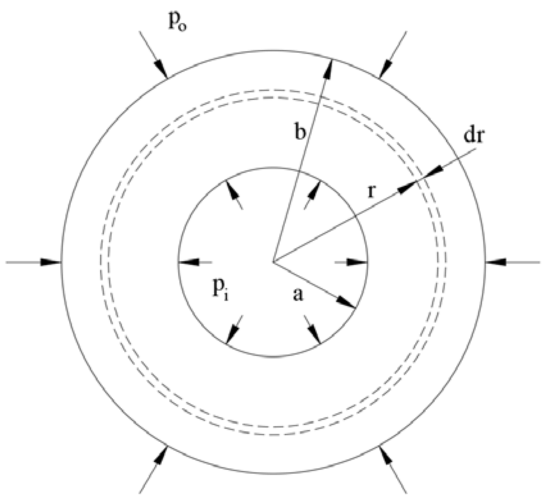


Figure 7 Internal and external pressure on a thick-walled cylinder

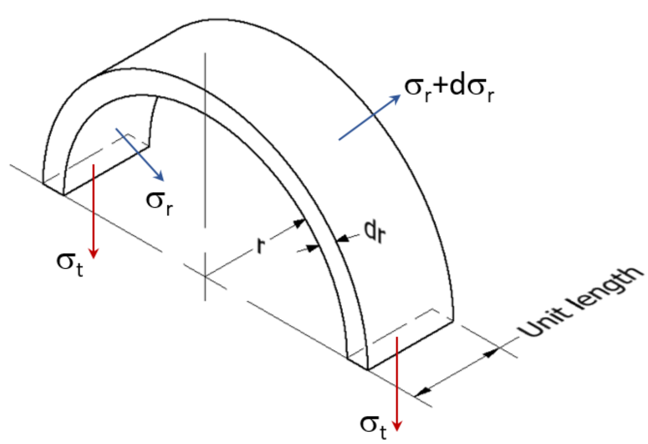


Figure 8 Stresses on the half shell

The equation of equilibrium can be developed using cylindrical coordinates [note that cylindrical coordinate systems will use θ in place of *t* for the tangential coordinate, but *t* will be used here to stay consistent with the thin-walled pressure vessel equations]. Without proof, the equation of equilibrium reduces to

$$r\frac{d}{dr}(\sigma_r)=\sigma_t-\sigma_r \qquad \text{Equation 8}$$

Similarly, the strain compatibility conditions are given by

$$r\frac{d}{dr}(\varepsilon_t)=\varepsilon_r-\varepsilon_\theta \qquad \text{Equation 9}$$

Additional equations come from Hook's law for triaxial stress. The common form of Hooke's law would be in rectangular coordinates.

$$\varepsilon_x=\frac{1}{E}\left[\sigma_x-\nu\left(\sigma_y+\sigma_z\right)\right]$$

$$\varepsilon_y=\frac{1}{E}\left[\sigma_y-\nu\left(\sigma_z+\sigma_x\right)\right]$$

$$\varepsilon_z=\frac{1}{E}\left[\sigma_z-\nu\left(\sigma_x+\sigma_y\right)\right]$$

where σ is stress, ε is strain, E is modulus of elasticity, and ν is Poisson's ratio. Hooke's law for triaxial stress can also be written in terms of cylindrical coordinates to give

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_t + \sigma_z\right)\right] \qquad \text{Equation 10}$$

$$\varepsilon_t = \frac{1}{E}\left[\sigma_t - \nu\left(\sigma_r + \sigma_z\right)\right] \qquad \text{Equation 11}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_r + \sigma_t\right)\right] \qquad \text{Equation 12}$$

From this point, there are multiple ways to develop the general equations for stress at any point. The process can become very tedious, and I will only provide a general approach to not overcomplicate the issue. As a basic approach, we can take Equation 12 and realize that E and ν are all constant material properties. Deformation at a cross-section sufficiently far from the ends will not vary [or vary insignificantly] in the *z* direction, so ε_z and σ_z are considered constant. Therefore, $\sigma_r + \sigma_t$ will also be constant. It is common to call the constant 2A (the factor of 2 is for simplification of future equations) giving

$$\sigma_r + \sigma_t = 2A \qquad \text{Equation 13}$$

Adding Equation 8 and Equation 13 gives

$$r\frac{d}{dr}\left(\sigma_r\right) + 2\sigma_r = 2A$$

$$r\frac{d}{dr}\left(\sigma_r\right) = 2\left(A - \sigma_r\right) \qquad \text{Equation 14}$$

Equation 14 is a first order differential equation with a single variable σ_r, which can be solved using separation of variables to give

$$\sigma_r = A - \frac{B}{r^2} \qquad \text{Equation 15}$$

where A and B are constants. Substituting Equation 15 into Equation 13 gives

$$\sigma_t = A + \frac{B}{r^2} \qquad \text{Equation 16}$$

The constants A and B can be determined by using boundary conditions for radial stress, which must equal the internal pressure at the inner radius and the external pressure at the outer radius (both negative). Determining the constants will give the final form of the equations shown in Equation 17 and Equation 18.

Radial Stress at any point $$\sigma_r = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} - \frac{a^2 b^2 (p_i - p_o)}{(b^2 - a^2) r^2} \qquad \text{Equation 17}$$

Tangential Stress at any point $$\sigma_t = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} + \frac{a^2 b^2 (p_i - p_o)}{(b^2 - a^2) r^2} \qquad \text{Equation 18}$$

For the case of internal pressure and external pressure, the radial displacement at any radial location is given by

$$u = \frac{r}{E(b^2 - a^2)}\left[(1-\nu)(p_i a^2 - p_o b^2) + \frac{(1+\nu) a^2 b^2}{r^2}(p_i - p_o)\right] \qquad \text{Equation 19}$$

Example 4

A thick-walled cylinder has an inside radius of 1.25 inches and an outside radius of 3.25 inches. The outside pressure is 100 psi and the internal pressure is 1300 psi. Plot the distribution of radial stress over the thickness.

Solution:

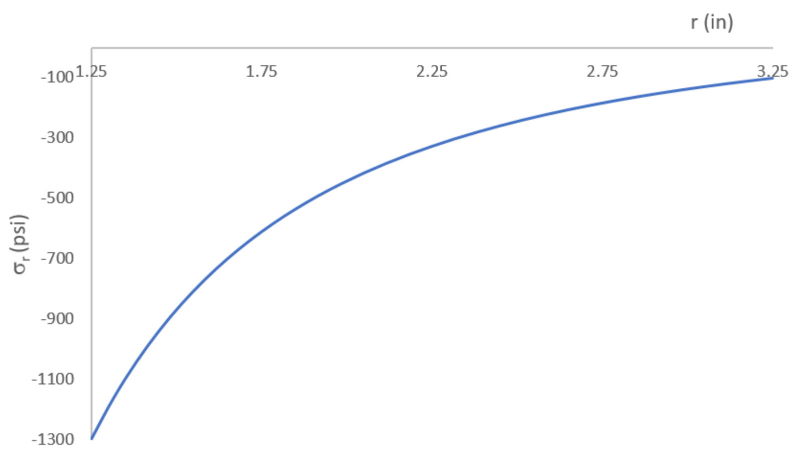
The radial stress can be calculated using Equation 17.

$$\sigma_r = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} - \frac{a^2 b^2 (p_i - p_o)}{(b^2 - a^2) r^2}$$

$$\sigma_r = \frac{1.25^2 (1300) - 3.25^2 (100)}{3.25^2 - 1.25^2} - \frac{1.25^2 (3.25^2)(1300 - 100)}{(3.25^2 - 1.25^2) r^2}$$

$$\sigma_r = 108.33 - \frac{19804.69}{9 r^2}$$

Plotting the equation for values of *r* between 1.25 and 3.25 gives



Note that the radial stress equals the internal pressure at *a* and the external pressure at *b* (all negative).

3.3 Internal Pressure Only

The equations can be simplified for specific loading cases. If the pressure vessel only has internal pressure $(p_o = 0)$, Equation 17 and Equation 18 reduce to

Radial Stress $$\sigma_r = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$ Equation 20

Tangential Stress $$\sigma_t = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$ Equation 21

Figure 9 provides a graphical representation of the stresses. The radial stress distribution calculated from Equation 20 is shown in Figure 9 (a). The radial stresses are negative (compression) with a maximum value at the inner surface. Figure 9 (b) shows the tangential stress distribution calculated from Equation 21. The stresses are positive (tensile) with a maximum value at the inner surface.

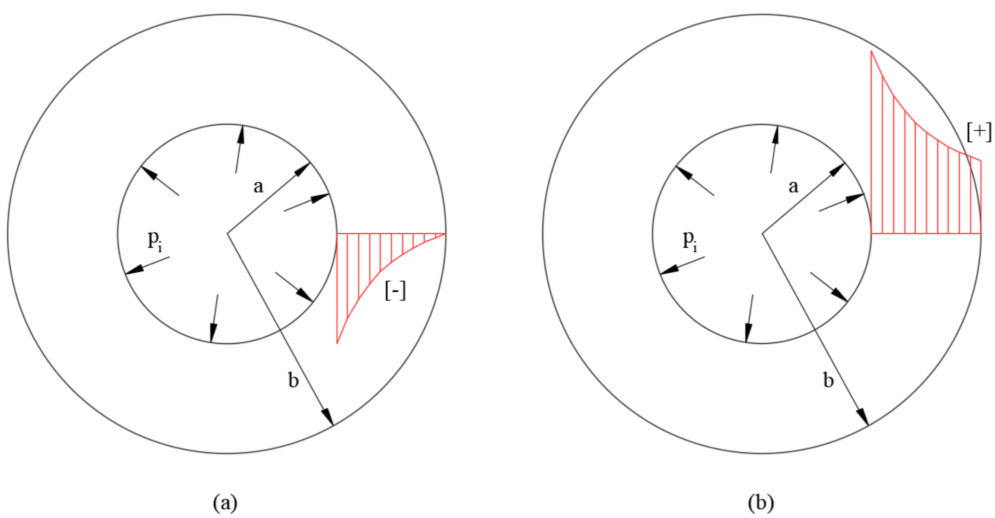


Figure 9 (a) Radial stress distribution and (b) tangential stress distribution for internal pressure only

The radial stress is always compressive and the tangential stress is always a tensile stress, with the tangential stress larger than the radial. The maximum radial stress occurs at the inside surface $(r = a)$ giving

$$(\sigma_r)_{\max} = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{a^2}\right)$$

$$(\sigma_r)_{\max} = \left(\frac{a^2 - b^2}{b^2 - a^2}\right) p_i$$

Maximum Radial Stress $$(\sigma_r)_{\max} = -p_i$$ Equation 22

Similarly, the maximum tangential stress occurs at the inner surface.

$$(\sigma_t)_{\max} = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{a^2}\right)$$

Maximum Tangential Stress $$(\sigma_t)_{\max} = \left(\frac{a^2 + b^2}{b^2 - a^2}\right) p_i$$ Equation 23

For the case of internal pressure only, the radial displacement at the inner surface is

$$u_a = \frac{a p_i}{E}\left(\frac{a^2 + b^2}{b^2 - a^2} + \nu\right)$$ Equation 24

Example 5

A thick-walled cylinder has an inside radius of 6 inches and an outside radius of 10 inches. Plot the distribution of tangential stress for an inside pressure of 9 ksi. What is the maximum tangential stress?

Solution:

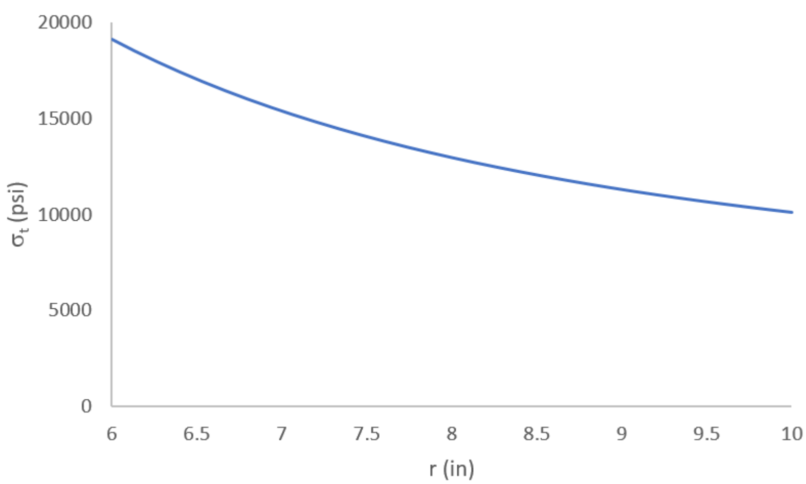
The tangential stress is calculated using Equation 21.

$$\sigma_t = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$\sigma_t = \frac{6^2(9000)}{10^2 - 6^2}\left(1 + \frac{10^2}{r^2}\right)$$

$$\sigma_t = 5062.5\left(1 + \frac{100}{r^2}\right) \quad [psi]$$

Plotting the equation for values of r ranging from 6 inches to 10 inches gives



The maximum tangential stress can be calculated using Equation 23 (or from the above equation with r = 6 inches).

$$(\sigma_t)_{\max} = \left(\frac{a^2+b^2}{b^2-a^2}\right)p_i$$

$$(\sigma_t)_{\max} = \left(\frac{6^2+10^2}{10^2-6^2}\right)9000$$

$$(\sigma_t)_{\max} = 19{,}125\,psi$$

Example 6

The pressure vessel in the previous example is made from steel with a modulus of elasticity of 29x10^6 psi and has a Poisson's ratio of 0.3. Find the radial displacement at the inner surface in inches.

Solution:

The radial displacement is determined using Equation 24.

$$u_a = \frac{ap_i}{E}\left(\frac{a^2+b^2}{b^2-a^2}+\nu\right)$$

$$u_a = \frac{6(9000)}{29\times10^6}\left(\frac{6^2+10^2}{10^2-6^2}+0.3\right)$$

$$u_a = 0.0045in$$

3.4 External Pressure Only

If the pressure vessel only has external pressure $(p_i = 0)$, Equation 17 and Equation 18 reduce to

$$\sigma_r = \frac{-b^2 p_o}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right) \qquad \text{Equation 25}$$

$$\sigma_t = \frac{-b^2 p_o}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) \qquad \text{Equation 26}$$

Equation 25 and Equation 26 are graphically represented in Figure 10 (a) and Figure 10 (b) respectively. The radial stresses are negative (compression) with a maximum value, equal to the magnitude of the external pressure, at the outer surface. The tangential stresses are compression with the maximum value at the inner surface.

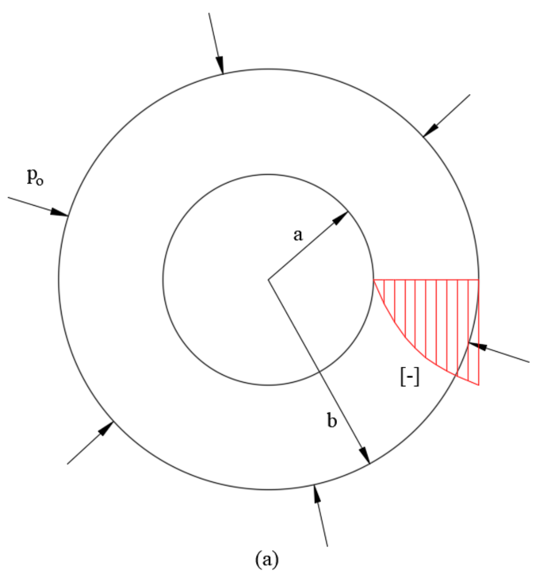


Figure 10 (a) Radial stress distribution and (b) tangential stress distribution for external pressure only

The maximum radial stress occurs at the outer surface $(r = b)$.

$$\sigma_r = \frac{-b^2 p_o}{b^2 - a^2}\left(1 - \frac{a^2}{b^2}\right)$$

$$\sigma_r = \frac{-b^2 p_o}{b^2 - a^2} + \frac{-a^2 p_o}{b^2 - a^2}$$

Maximum Radial Stress

$$\sigma_r = -p_o \qquad \text{Equation 27}$$

The maximum tangential stress occurs at the inner surface $(r = a)$ giving

$$\sigma_t = \frac{-b^2 p_o}{b^2 - a^2}\left(1 + \frac{a^2}{a^2}\right)$$

Maximum Tangential Stress

$$\sigma_t = \frac{-2b^2 p_o}{b^2 - a^2} \qquad \text{Equation 28}$$

For the case of external pressure only, the radial displacement at the outer surface is

$$u_b = -\frac{bp_o}{E}\left(\frac{a^2 + b^2}{b^2 - a^2} + \nu\right) \qquad \text{Equation 29}$$

Example 7

A thick-walled pressure vessel has an outside diameter of 16 inches and an inside diameter of 12 inches. The vessel is subjected to an external pressure only, and the maximum radial stress has a magnitude of 4 ksi. The material properties are $E = 29 \times 10^6\ psi$ and $\nu = 0.28$. What is the maximum tangential stress and radial displacement at the outer surface?

Solution:

Based on Equation 27, the external pressure will equal the magnitude of the maximum radial stress. Therefore,

$$p_o = |\sigma_r| = 4{,}000\, psi$$

Equation 28 gives

$$\sigma_t = \frac{-2b^2 p_o}{b^2 - a^2} = \frac{-2(8^2)(4000)}{8^2 - 6^2}$$

$$\sigma_t = 18{,}286\, psi$$

Equation 29 gives the radial displacement.

$$u_b = -\frac{bp_o}{E}\left(\frac{a^2 + b^2}{b^2 - a^2} + \nu\right)$$

$$u_b = -\frac{8(4000)}{29 \times 10^6}\left(\frac{6^2 + 8^2}{8^2 - 6^2} + 0.28\right)$$

$$u_b = -0.00425 in$$

4.0 Failure Criteria

A complete coverage of failure criteria is beyond the scope of this course. However, a general coverage will be provided with the focus on failure of thick-walled pressure vessels. Many different failure theories exist, but they can be separated into two major categories based on material type. The categories are failure theories for brittle materials and failure theories for ductile materials. Many theories exist for each material category, but the most common will be discussed here.

4.1 Brittle Materials

For brittle materials, failure is specified by fracture. The two most common theories of failure for brittle materials are described below. The maximum stress theory should be used if the material has similar behavior in tension and compression. If the material behavior is different in tension and compression, then Mohr's failure criterion should be used.

4.1.1 Maximum Stress Theory

The maximum stress theory is based on the assumption that failure occurs when the maximum principal stress reaches a limit value such as ultimate strength.

$$|\sigma_1| = \sigma_{ult}$$
$$|\sigma_2| = \sigma_{ult}$$

Equation 30

where σ_1 and σ_2 are the principal stresses and σ_{ult} is the ultimate strength. Equation 30 is shown graphically in Figure 11 (a). If the stress at a point (σ_1, σ_2) falls outside the shaded region or on the boundary, the material will fracture.

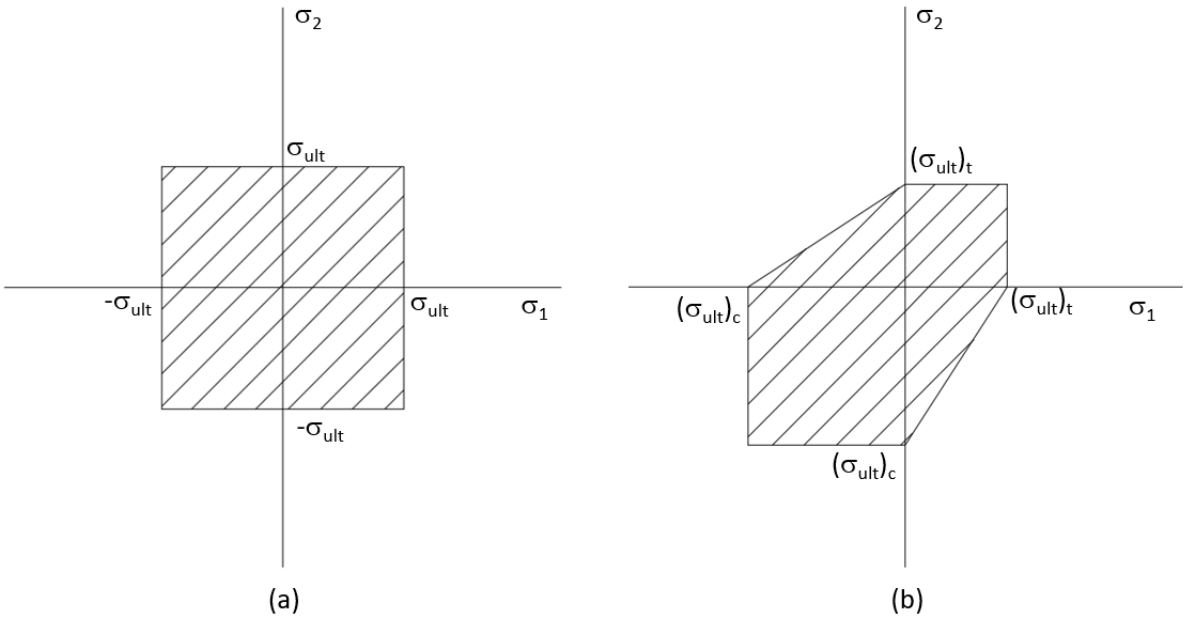


Figure 11 Brittle material theories (a) Maximum stress theory (b) Mohr's failure criterion

4.1.2 Mohr's Failure Criterion

If the material exhibits different properties in tension and compression, a modified plot will exist as shown in Figure 11 (b). The only difference is that the ultimate strength in tension is not the same as that in compression. The same principle applies, that stress inside the shaded region is considered safe.

4.2 Ductile Materials

For ductile materials, failure is specified by the initiation of yielding. Failure will occur due to shear stress. The two most common theories for ductile materials are described below, with the maximum shear theory being more conservative.

4.2.1 Maximum Shear Theory

The maximum shear theory assumes that yielding will begin when the maximum shearing stress is equal to the that of simple tension. The maximum shear stress can be determined from equations of Mohr's circle to be one half the difference between the principal stresses.

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} \qquad \text{Equation 31}$$

Let's look at the case of a thick-walled pressure vessel with internal pressure only and σ_z is equal to zero. The maximum shear stress will occur at the inner surface and will have a magnitude of

$$\tau_{\max} = \frac{(\sigma_t)_{\max} - (\sigma_r)_{\max}}{2} \qquad \text{Equation 32}$$

From Equation 22 and Equation 23, noting that tangential stress is positive and radial stress is negative, the maximum shear stress becomes

$$\tau_{\max} = \frac{\left(\frac{a^2 + b^2}{b^2 - a^2}\right)p_i - \left(\frac{a^2 - b^2}{b^2 - a^2}\right)p_i}{2}$$

Maximum Shear Stress: Internal Pressure Only

$$\tau_{\max} = \left(\frac{b^2}{b^2 - a^2}\right)p_i \qquad \text{Equation 33}$$

Based on the idea that ductile materials fail in shear, the maximum shear theory of failure can be written as

$$\left.\begin{aligned} |\sigma_1| &= \sigma_Y \\ |\sigma_2| &= \sigma_Y \end{aligned}\right\} \text{ If } \sigma_1 \text{ and } \sigma_2 \text{ have the same sign}$$

Equation 34

$$|\sigma_1 - \sigma_2| = \sigma_Y \quad \text{If } \sigma_1 \text{ and } \sigma_2 \text{ have the opposite signs}$$

where σ_1 and σ_2 are the principal stresses and σ_Y is the yield strength. Equation 34 is shown graphically in Figure 12 as the solid line shaded region. If the stress at a point (σ_1, σ_2) falls outside the shaded region or on the boundary, the material will yield.

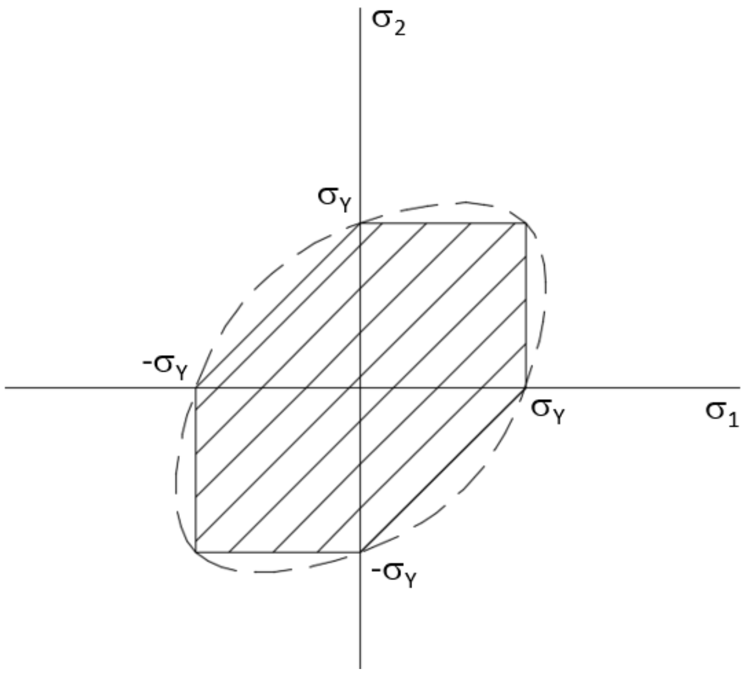


Figure 12 Ductile material theories: Maximum shear theory (solid line shaded region) and Von Mises yield theory (dashed line)

4.2.2 Von Mises Yield Theory

The last failure theory is based on distortional energy. The Von Mises yield theory is formulated based on distortions caused by strain energy. Without proof, the theory states that failure occurs when

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2 \qquad \text{Equation 35}$$

Equation 35 is shown graphically in Figure 12 as the dashed line elliptical curve region. If the stress at a point (σ_1, σ_2) falls outside the shaded region or on the boundary, the material will yield. It can be seen that the maximum shear theory is more conservative than the Von Mises yield theory.

Example 8

A thick-walled pressure vessel has an inner radius of 5 inches and an outer radius of 6 inches. The vessel is subjected to an internal pressure of 3500 psi. Based on the Von Mises yield theory, determine if failure occurs using a material yield strength of 36 ksi.

Solution:

The maximum radial stress and maximum tangential stress both occur at the inner radius, so that will be the location of the maximum Von Mises stress. From Equation 22

$$(\sigma_r)_{\max} = -p_i = -3500\frac{lb}{in^2}$$

From Equation 23

$$(\sigma_t)_{\max} = \left(\frac{a^2 + b^2}{b^2 - a^2}\right) p_i = \left(\frac{5^2 + 6^2}{6^2 - 5^2}\right)(3500) = 19{,}409 \frac{lb}{in^2}$$

Because the stress element on the inner surface will not have any shear stress loading, the radial and tangential stress values will also be the principal stress values.

$$\sigma_1 = -3{,}500 \frac{lb}{in^2}$$

$$\sigma_2 = 19{,}409 \frac{lb}{in^2}$$

Based on Equation 35, failure occurs if

$$\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_Y^2$$

$$(-3500)^2 - -(3500)(19409) + 19409^2 = 36000^2$$

$$21375 < 36000$$

Therefore, failure does not occur based on the Von Mises yield theory

