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# Feedback Control System Fundamentals

by

Peter Kennedy



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1.0 Introduction: This course discusses many fundamental concepts associated with classical feedback control theory. Feedback, in its most general sense, refers to the interaction between interconnected dynamic systems such that the response of one influences the dynamics of the other. In the context of feedback control; the state of a physical system or device is measured by a sensing system. The measured state or feedback is compared to a desired state and the error used by a controller to reduce the difference between the actual and desired states. Using the difference or error, equates to a negative feedback control loop; central to feedback control theory.

The physical device or system to be controlled is often called the plant, process, or load. An example of a feedback control system is the central heating and air conditioning system for a home, or building. A thermostat or temperature sensor is the feedback sensor that measures the room temperature and compares it to the desired temperature or set point, calculating a difference or error. If the temperature is less than the set point, the error is used by the controller to force more heat into the room. When the set point is reached, the error is zero or below an error threshold and the controller will stop heating the room. Another example is the speed control in most of today's automobiles. The speed of the vehicle is measured and compared to a desired speed. Based on the difference between actual speed and the set point, acceleration or braking is applied to the automobile drive to null the error and maintain the desired speed.

Classical control deals directly with the differential equations that describe the dynamics of a plant or process. These equations are transformed into frequency dependent transfer functions. The transfer function is the ratio of two frequency dependent polynomials whose roots describe the response of the plant in a frequency domain. The controller or compensator shapes the closed feedback loop response, given the plant response, to achieve the control performance objectives. Modern control theory is another approach to control loop design and analysis; especially useful for systems with multiple inputs and outputs. It utilizes State Space methods to evaluate the response of a system as well as generate the control for it. This relies heavily on linear algebra and matrix theory. Modern control theory and design methods are not discussed further in this course. Even classical feedback control design and analysis tends to require a good foundation in mathematics, however the purpose of this course is not to dwell on the math, although examples are provided, but to provide the basic design and analysis concepts.

2.0 Feedback Control Block Diagram: The feedback control system block diagram provides both a visual as well as a basis for the mathematical representation of the feedback control system. It



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is an important part of the control system design and analysis. In general, the feedback control block diagram will be complex incorporating many elements and several feedback loops. To describe just the basic elements, a very simple block diagram is shown in Figure 1.

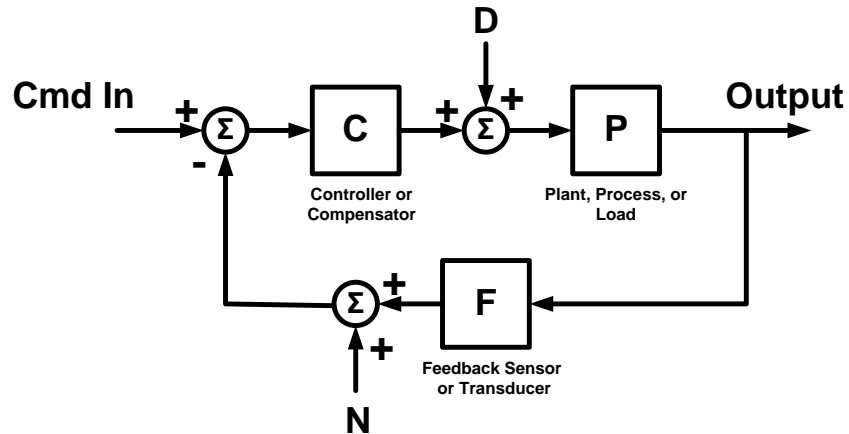


Figure 1 Basic Feedback Control Block Diagram

This simple loop includes a controller (C), plant (P), and feedback sensor (F). The inputs are the command input (Cmd In), disturbances (D) and sensor noise (N). The command, which is the desired plant output, is applied to the input summing junction and compared with the actual plant output. The error between the command and actual output is calculated and fed to the controller. The controller generates the drive signal to the plant that reduces the error until the command input equals the actual output or remains within a desired offset.

The dynamics of each block can be expressed in the time or frequency domain. The time domain is very useful for simulating the loop response and evaluating performance as a function of time. A set of differential equations which can include non-linear terms are used to describe the dynamics for each block. The frequency domain is excellent for linear analysis. The gain of each element can be characterized as a function of frequency; effectively each element is a frequency dependent gain. The output response can be evaluated as a function of frequency and the controller adjusted in the frequency domain to improve the loop performance.

3.0 Time and Frequency Domain Representations of Loop Dynamics: The plant dynamics is normally described by a set of differential equations and non-linearity's (time and/or multi-variable dependent). To describe the response in the frequency domain, the Laplace transform or the differential operator can be used if the plant is linear or approximated by linearizing the system about an operating point. The definition of the Laplace transform of a time dependent variable  $f(t)$  is defined as:



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$$F(s) = \int_0^{\infty} f(t)e^{-s \cdot t} dt$$

This integral exists for an  $s = \sigma + j\omega$  with any real part  $\sigma > 0$ . The variable  $\omega = 2\pi f$  where 'f' is frequency in hertz. Differentiating this expression n times with respect to time, results in the important relationship:

$$\int_0^{\infty} f^n(t)e^{-s \cdot t} dt = s^n F(s) - \sum_{k=1}^n s^{k-1} \cdot f^{n-k}(0)$$

A differentiated function in time is algebraically related to its transform as  $L[f^n(t)] \rightarrow s^n F(s)$  where 'L' denotes the Laplace transform. The summation term on the right accounts for the initial conditions of each differentiated term. For frequency analysis, the initial conditions are often assumed to be zero so that:

$$\int_0^{\infty} f^n(t)e^{-s \cdot t} dt = s^n F(s) \quad \text{for initial conditions} = 0$$

The inverse Laplace transform is used to convert a frequency domain function back to one in the time domain. The inverse transform is given by:

$$f(t) = \frac{1}{2 \cdot \pi \cdot j} \int_{c-j\infty}^{c+j\infty} F(s)e^{s \cdot t} ds$$

This complex integral is evaluated along the path  $s = c + j\omega$  in the complex plane from  $c - j\infty$  to  $c + j\infty$  where c is any real number for which the path  $c + j\omega$  lies in the region of convergence of the transform  $F(s)$ . Although this is a somewhat complicated definition; methods such as partial fraction expansion are available for evaluating the inverse transform. Most control textbooks and engineering reference books have very thorough tables of both the Laplace transform and the inverse Laplace transform solved for numerous functions. So in many cases, one can obtain the transform or its inverse from a table and need not evaluate it directly.

It is also important to note that although the definition of the Laplace variable is  $s = \sigma + j\omega$ , when performing frequency domain analysis the variable will be equated to  $s = j\omega$  where  $\omega = 2\pi f$ . The  $\sigma$  term is assumed zero since for frequency domain analysis a function is being evaluated based upon its response to periodic sinusoidal signals. The real axis ( $\sigma$ ) in the s-domain represents an exponential decay or growth factor that is not relevant for this analysis.

The relationship between the time and frequency domain can be shown, using the Laplace transform, for a first and second order plant as follows. For a first order plant (P), let  $x(t)$  equal the plant input and to shorten notation let  $y(t)$  equal Output(t), then the plant can be described by the differential equation:



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$$\dot{y}(t) + \tau \cdot y(t) = \tau \cdot K_S \cdot x(t)$$

The term  $\tau$  is the plant time constant while  $K_S$  is the scaling factor. Taking the Laplace transform, this converts in the frequency domain (denoted by the Laplace variable 's') to:

$$(s + \tau) \cdot Y(s) = \tau \cdot K_S \cdot X(s)$$

or

$$\text{Output}(s) = Y(s) = P(s) \cdot X(s) = \frac{\tau \cdot K_S}{s + \tau} \cdot X(s)$$

$$\text{where } P(s) = \frac{\tau \cdot K_S}{s + \tau}$$

$$\text{and } L(y(t)) = Y(s) \quad ; \quad L(\dot{y}(t)) = s \cdot Y(s) \quad ; \quad L(x(t)) = X(s)$$

A second order system can be expressed as:

$$\ddot{y}(t) + 2 \cdot \xi \cdot \omega_0 \cdot \dot{y}(t) + \omega_0^2 \cdot y(t) = \omega_0^2 \cdot x(t)$$

For this plant  $\xi$  is the damping constant and  $\omega_0$  the natural frequency of the plant. The Laplace transform of this equation is:

$$(s^2 + 2 \cdot \xi \cdot \omega_0 \cdot s + \omega_0^2) \cdot Y(s) = \omega_0^2 \cdot X(s)$$

or

$$\text{Output}(s) = Y(s) = P(s) \cdot X(s) = \frac{\omega_0^2}{s^2 + 2 \cdot \xi \cdot \omega_0 \cdot s + \omega_0^2} \cdot X(s)$$

$$\text{where } P(s) = \frac{\omega_0^2}{s^2 + 2 \cdot \xi \cdot \omega_0 \cdot s + \omega_0^2}$$

$$\text{and } L(\ddot{y}(t)) = s^2 \cdot Y(s)$$

This general procedure can be applied to a differential equation of any order.

**4.0 Key Feedback Loop Relationships:** Each block in the feedback control loop block diagram can be expressed in the frequency domain using the Laplace transform described in the last section, assuming all constraints are met. In the frequency domain, blocks can be treated algebraically with each element of the loop in the block diagram treated as a frequency dependent function. Referring to the block diagram of Figure 1, an expression for the Output can be obtained algebraically as:

$$\text{Output}(s) = P(s) \cdot [D(s) + C(s) \cdot (\text{Cmd } In(s) - (N(s) + F(s) \cdot \text{Output}(s)))]$$

Solving for the output:



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$$Output(s) = \left[ \frac{P(s) \cdot C(s)}{1 + P(s) \cdot C(s) \cdot F(s)} \right] \cdot Cmd\ In(s) + \left[ \frac{P(s)}{1 + P(s) \cdot C(s) \cdot F(s)} \right] \cdot D(s) - \left[ \frac{P(s) \cdot C(s)}{1 + P(s) \cdot C(s) \cdot F(s)} \right] \cdot N(s)$$

The terms in brackets that multiply each input are very important relative to the control loop design. The first term in brackets describes the response from the command input to the output and is termed the closed loop transfer function (CLTF) with a magnitude called the closed loop gain (CLG) or:

$$CLTF(s) = \left[ \frac{P(s) \cdot C(s)}{1 + P(s) \cdot C(s) \cdot F(s)} \right]$$
$$CLG(s) = \left[ \left[ \frac{P(s) \cdot C(s)}{1 + P(s) \cdot C(s) \cdot F(s)} \right] \right]$$

The control compensator gain,  $|C(s)|$ , is normally a very high such that magnitude of the product of the three terms  $|P(s) \cdot C(s) \cdot F(s)| \gg 1$  over the effective operating bandwidth of the loop. This product is the open loop transfer function (OLTF) and its magnitude called the open loop gain (OLG).

$$OLTF(s) = P(s) \cdot C(s) \cdot F(s)$$
$$OLG(s) = |P(s) \cdot C(s) \cdot F(s)|$$

The feedback term is often a unity gain (or scaled to provide unity gain in the feedback path) as it is simply sensing the plant output. This being the case, the closed loop gain, CLG, is unity over the operating bandwidth of the loop and rolls off towards zero at high frequency. If the feedback gain is not unity, it will change the closed loop gain from one to the inverse of the feedback gain magnitude (i.e.  $1/F$ ).

In the expression for the loop output above, the second term accounts for the impact of disturbances on the output response. The transfer function in brackets multiplying the disturbance is often termed the compliance, as in how compliant the loop is to a disturbance.

$$Compliance(s) = \left[ \frac{P(s)}{1 + P(s) \cdot C(s) \cdot F(s)} \right]$$

This term attenuates the disturbance primarily due to the high gain control compensator,  $C(s)$ , in the open loop gain. Therefore the higher the compensator gain, the more disturbance rejection and the less impact the disturbances will have on the output. However this gain will be limited by the stability of the loop. Finally, as modeled, the last term is sensor noise which as shown effectively adds directly to the output since it is multiplied by the closed loop gain. As it is basically input to the same summing junction as the command input, this would be expected.



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The error between the command input and the output is given by:

$$Error(s) = Cmd\ In(s) - Output(s) = \left[ \frac{1 - P(s) \cdot C(s)(1 - F(s))}{1 + P(s) \cdot C(s) \cdot F(s)} \right] \cdot Cmd\ In(s) - \left[ \frac{P(s)}{1 + P(s) \cdot C(s) \cdot F(s)} \right] \cdot D(s) + \left[ \frac{P(s) \cdot C(s)}{1 + P(s) \cdot C(s) \cdot F(s)} \right] \cdot N(s)$$

The error is effectively the accuracy with which the output follows the command input. The first term in brackets that is multiplying the command input is often called the loop sensitivity function. For unity feedback ( $F(s) \sim 1$ ) this term is effectively one over the open loop gain, similar to the compliance function discussed previously, or:

$$Sensitivity(s) = \left[ \frac{1}{1 + P(s) \cdot C(s)} \right] \quad \text{for } F(s) = 1$$

The error will be a function of the attenuation provided by this term as well as the magnitude of the command, disturbance, and noise. One final error relationship of importance is the steady state error which is defined as:

$$E_{SS} = \lim_{s \rightarrow 0} [s \cdot Error(s)] = \lim_{s \rightarrow 0} \left[ \left( \frac{s}{1 + P(s) \cdot C(s)} \right) \cdot Cmd\ In(s) \right]$$

It is a measure of the error response or effectively the accuracy of a stable system as time goes to infinity. It is normally evaluated against three standard input signals; a step, ramp or rate, and parabolic input or acceleration. These signals have the Laplace transforms:

$$\text{step} \rightarrow \frac{K_{step}}{s} \quad ; \quad \text{ramp} \rightarrow \frac{K_{ramp}}{s^2} \quad ; \quad \text{parabolic} \rightarrow \frac{K_{parabolic}}{s^3}$$

The steady state error is often used as part of the control loop performance specification. The required response to these input signals will determine what is termed the loop ‘Type’; which refers to the open loop poles at  $s=0$  or the number of integrators (i.e.  $1/s$  terms) in the OLTF(s). If we describe a general input to the system as  $K_x/s^n$  which can be any of the above inputs depending on the value of  $n$ , then the steady state error can be written as:

$$E_{SS} = \lim_{s \rightarrow 0} [s \cdot Error(s)] = \lim_{s \rightarrow 0} \left[ \left( \frac{s}{1 + P(s) \cdot C(s)} \right) \cdot \frac{K_x}{s^n} \right] = \lim_{s \rightarrow 0} \left[ \left( \frac{K_x}{s^{n-1} + s^{n-1} \cdot P(s) \cdot C(s)} \right) \right]$$

From this expression, the following table can be generated that describes the steady state error as a function of loop type and input.



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Table 1 Steady State Error versus Loop Type and Input

Input	Steady State Error			
	Type 0 System (no open loop poles at s=0)	Type 1 System (one open loop pole at s=0)	Type 2 System (two open loop poles at s=0)	More than two open loop poles at s=0
Step (n=1)	$\frac{K_{step}}{1 + K_p}$	0	0	0
Ramp (n=2)	$\infty$	$\frac{K_{ramp}}{K_v}$	0	0
Parabolic (n=3)	$\infty$	$\infty$	$\frac{K_{parabolic}}{K_A}$	0
	$K_p = P(0) \cdot C(0)$	$K_v = \lim_{s \rightarrow 0} [s \cdot P(s) \cdot C(s)]$	$K_A = \lim_{s \rightarrow 0} [s^2 \cdot P(s) \cdot C(s)]$	

From Table 1 it is observed that with a Type 0 loop and a step input there will always be an offset between the input and output which is a function of the loop gain. The Type 0 loop cannot track a ramp or parabolic input as the error will continue to grow infinitely with time for both input types. A Type 1 loop will null the error to a step input at a rate based on the loop bandwidth. The Type 1 loop will track a ramp with an error that is a function of the loop gain. It cannot track a parabolic input as the error will continue to grow with time. The Type 2 loop tracks both a step and ramp with zero error and a parabolic input with an offset between the input and output which is a function of the loop gain. Loop Types greater than two can track all three inputs with zero error; however a loop with this many integration terms is difficult to stabilize.

5.0 Control Loop Stability: There are many methods for analyzing feedback control loop stability. For a linear system, if the feedback control loop output remains bounded for any bounded input (Cmd In, D, N), then the system is considered to be bounded input- bounded output stable. For non-linear systems, more advanced methods such as the Lyapunov criteria are required.





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Evaluating stability for linear systems can be performed easily in the frequency domain. The primary transfer functions C, P, F are described in the frequency domain by the ratio of two polynomials that are a function of 's'. The roots of these polynomials in the numerator are termed zeros and in the denominator termed poles. The closed and open loop transfer function gain and phase is a function of all these polynomials. For example if subscript 'N' denotes numerator and 'D' the denominator then the closed loop transfer function can be expressed as:

$$CLTF(s) = \left[ \frac{P_N(s) \cdot C_N(s) \cdot F_D(s)}{P_D(s) \cdot C_D(s) \cdot F_D(s) + P_N(s) \cdot C_N(s) \cdot F_N(s)} \right]$$

$$CLTF_N(s) = P_N(s) \cdot C_N(s) \cdot F_D(s)$$

$$CLTF_D(s) = P_D(s)C_D(s)F_D(s) + P_N(s)C_N(s)F_N(s)$$

One important observation is that the zeros of the OLTF will also be the zeros of the CLTF. For a causal linear system to be stable all of the poles of CLTF<sub>D</sub>(s) must have negative-real values. Determining the poles of the polynomial can be done with a root finder. A stability method called the Routh criterion, based upon evaluating only combinations of the polynomial coefficients can also be used to determine if the poles all have negative real parts. However this does not really quantify how stable the system is or the margin of stability. Other methods using a frequency domain representation to evaluate stability while also providing stability margins are: the Bode Plots, Nyquist Plots, and Nichols plots.

5.1 Bode Stability Criterion: The Bode Plot is a relatively straightforward graphical method using plots of the OLTF gain and magnitude to evaluate stability. The Bode criteria for stability, however, is sufficient only for feedback loops whose CLTF is minimum phase (all zero's and poles of CLTF have negative real parts or are in left half of s-plane). The Bode plot is effectively two plots, one of the open loop gain (OLG) or magnitude and the other the phase of the OLTF. The gain and phase of an arbitrary transfer function G(s) are determined by substituting s=jω into the numerator polynomial, G<sub>N</sub>(jω), and the denominator polynomial, G<sub>D</sub>(jω), of G(jω) and then calculating the real and imaginary components of the numerator and denominator or:

$$G(s)_{s=j\omega} = \frac{G_N(j\omega)}{G_D(j\omega)} = \frac{RG_N + j \cdot IG_N}{RG_D + j \cdot IG_D}$$

where  $RG_N = \text{Real}[G_N(j\omega)]$  ;  $IG_N = \text{Imaginary}(G_N(j\omega))$   
 $RG_D = \text{Real}[G_D(j\omega)]$  ;  $IG_D = \text{Imaginary}(G_D(j\omega))$

The magnitude and phase are then determined as:



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$$|G(j\omega)| = \frac{\sqrt{RG^2 + IG^2}}{RG_D^2 + RI_D^2} \quad ; \quad \Phi(G(j\omega)) = \text{atan}\left(\frac{IG}{RG}\right) = \text{atan}\left(\frac{IG_N}{RG_N}\right) - \text{atan}\left(\frac{IG_D}{RG_D}\right)$$

$$\text{where } RG = RG_N \cdot RG_D + IG_N \cdot IG_D$$

$$IG = RG_D \cdot IG_N - IG_D \cdot RG_N$$

The magnitude of the product of several transfer functions is the product of their magnitudes and the phase is the sum of their phases or:

$$|G(j\omega)| = |G_1(j\omega) \cdot G_2(j\omega)| = |G_1(j\omega)| \cdot |G_2(j\omega)|$$

$$\Phi(G(j\omega)) = \Phi(G_1(j\omega) \cdot G_2(j\omega)) = \Phi(G_1(j\omega)) + \Phi(G_2(j\omega)) = \text{atan}\left(\frac{IG_1}{RG_1}\right) + \text{atan}\left(\frac{IG_2}{RG_2}\right)$$

The basic stability criterion can be determined by examining the denominator of the CLTF.

$$1 + P(s) \cdot C(s) \cdot F(s)$$

This function goes to infinity if

$$P(s) \cdot C(s) \cdot F(s) = -1$$

As the left side of this equation is the OLTF, instability will occur if the OLTF=-1 or

$$OLTF(s) = OLG(s) \cdot \Phi(OLTF(s)) = -1$$

$$\text{where } \Phi(OLTF(s)) - \text{phase of } OLTF$$

$$OLG(s) - \text{magnitude of } OLTF$$

This condition will occur if:

$$OLG(s) = 1 \quad \text{and} \quad \Phi(OLTF(s)) = 180^\circ$$

With a negative feedback loop, if OLTF=-1 this is equivalent to a signal input to the OLTF having the same amplitude and phase as is fed back; which is a condition for sustained oscillation.

Bode plots are used to determine just how close, or the margin the feedback loop has relative to an instability condition. The goal is to have sufficient margin to ensure the system remains stable. The Bode criteria defines two critical frequencies; the gain crossover frequency,  $f_{GC}$ , which is the frequency that the OLG crosses one and the phase crossover frequency,  $f_{PC}$ , which is the frequency at which the OLTF phase crosses  $-180^\circ$ . A stable phase margin is the amount that the OLTF phase angle is  $> -180^\circ$  when the OLG = 1 at the gain crossover frequency, or:

$$\text{Phase Margin} = PM = 180^\circ + \Phi(OLTF(f_{GC}))$$

$$\Phi(OLTF(f_{GC})) > -180^\circ \quad \text{or} \quad PM > 0 \quad \text{for stability}$$

The gain margin is the magnitude of the OLG relative to unity gain when the OLTF phase goes through  $-180^\circ$  at the phase crossover frequency or:

$$\text{Gain Margin} = GM = \frac{1}{OLG(f_{PC})}$$

$$OLG(f_{PC}) < 1 \quad \text{or} \quad GM > 1 \quad \text{for stability}$$



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The gain margin and phase margin are normally specified in the performance requirements for a control loop. The gain margin is often expressed in decibels, or:

$$GM_{dB} = 20 \cdot \text{Log}(GM) = -20 \cdot \text{Log}(\text{OLG}(f_{PC}))$$

It can be noted that given the CLTF satisfies the Bode criteria restrictions, the loop is stable if  $f_{GC} < f_{PC}$ . If the restrictions for the using the Bode criteria are not met, other methods such as the Nyquist or Nichols plots should be used. A couple of examples of the Bode plots follow. A plot is shown in Figure 2 for an OLTF given by:

$$\text{OLTF}(s) = \frac{1.6 \cdot (s + \omega_{z1})}{s \cdot (s + \omega_{p1}) \cdot (s^2 + 2 \cdot \xi \cdot \omega_r \cdot s + \omega_r^2)}$$

$$\text{where: } \omega_{z1} = 2 \cdot \pi \cdot f_{z1} = 2 \cdot \pi \cdot 0.1592$$

$$\omega_{p1} = 2 \cdot \pi \cdot f_{p1} = 2 \cdot \pi \cdot 0.01592$$

$$\omega_r = 2 \cdot \pi \cdot f_r = 2 \cdot \pi \cdot 0.6366 \quad ; \quad \xi = 0.25$$

The solid lines are for the open loop gain, which is scaled in units of dB on the right hand side of the plot and the dotted lines are for the open loop phase scaled in degrees on the left hand side of the plot. The plot of the OLG is in solid black with a line at 0 dB in solid light blue. Each pole contributes -20 dB/decade of slope to the magnitude plot and each zero +20 dB/decade of slope. The quadratic term contributes -40 dB/decade of slope. The OLTF phase is the dotted red line and a line at the -180° reference in dotted dark blue. Each pole contributes -90° of phase lag to the phase plot and each zero +90° of phase lead. The quadratic term contributes -180° of phase lag. As shown in the figure, the system has a gain crossover frequency at 0.05 Hz, a PM~30° and a GM~23 dB and is stable based upon the Bode stability criterion. The phase crossover frequency is 0.6°. This is a low bandwidth system and the relatively low PM indicating some oscillatory response can be expected.



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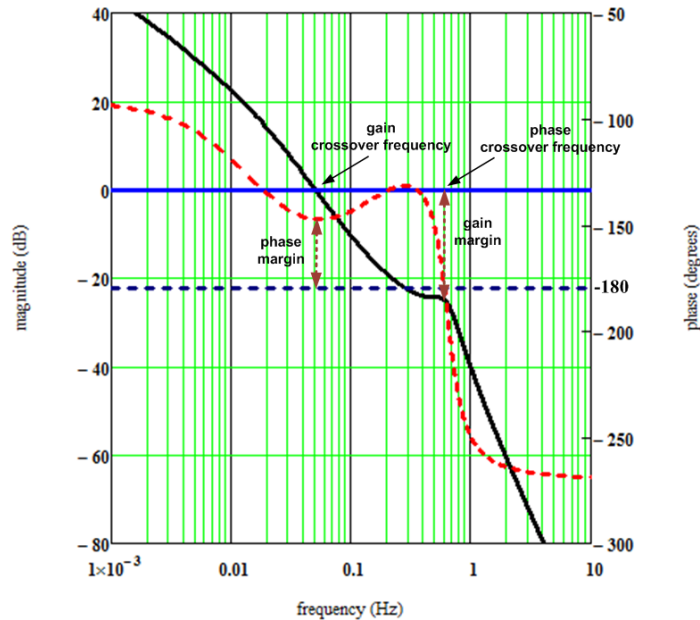


Figure 2 Bode Plot for Example 1

Another example is for a system with a much higher bandwidth system. Figure 3 provides a Bode plot for an OLF given by:

$$OLTF(s) = \frac{248040 \cdot (s + \omega_{z1})}{s^2 \cdot (s + \omega_{p1})}$$

$$\text{where: } \omega_{z1} = 2 \cdot \pi \cdot f_{z1} = 2 \cdot \pi \cdot 15.92$$

$$\omega_{p1} = 2 \cdot \pi \cdot f_{p1} = 2 \cdot \pi \cdot 159.2$$

Again the plot of the OLG is in solid black with a line at 0 dB in solid light blue. The OLF phase is the dotted red line and a line at the -180° reference in dotted dark blue (at the bottom of the plot). The system is stable with gain crossover frequency at 40 Hz, a PM~55° and a gain margin that is infinite (phase is always greater than and never crosses -180°). The PM for this system is better than the previous example and in the range that indicates a good response.



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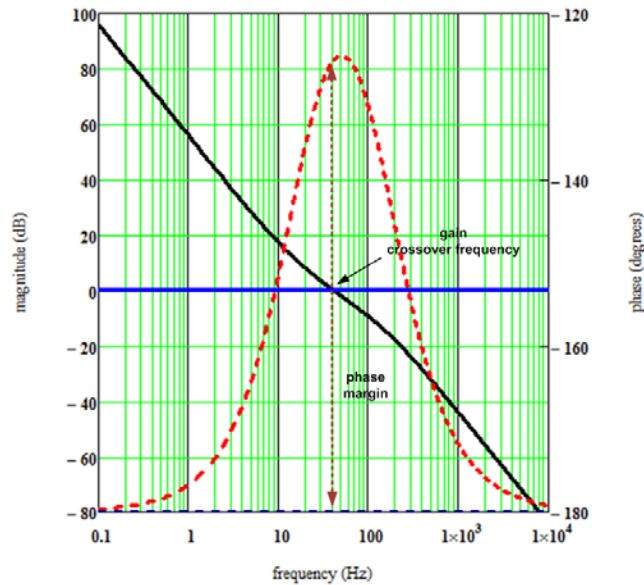


Figure 3 Bode Plot for Example 2

5.2 Nyquist Stability Criterion: The Nyquist stability criterion is another graphical technique for determining the system stability. The Nyquist plot is a graph of the OLTF imaginary versus real parts as a function of  $s=j\omega$ . The real part is plotted on x-axis and the imaginary on the y-axis. The plot can also be implemented as a polar plot of the OLTF magnitude and phase as a function of  $s=j\omega$ . It can be applied without explicitly computing the poles and zeros of either the closed-loop or open-loop system. As a result, it is applicable to systems defined by non-rational functions and delays. In contrast to Bode plots, it can handle minimum phase transfer functions as well as those with right half-plane singularities, or non-minimum phase plants. Although it is one of the more generic stability tests, it is still restricted to linear, time-invariant systems. Non-linear systems must use more complex stability criteria, such as Lyapunov or the circle criterion. While Nyquist is a graphical technique, it provides only a limited amount of information about the reason a system is stable or unstable, or how to modify an unstable system to be stable. Techniques like the Bode plots previously discussed, while less general, are sometimes a more useful design tool.

Assessment of the stability of a closed-loop negative feedback system is done by applying the Nyquist stability criterion to the Nyquist plot. The plot is effectively a mapping of a contour with infinite radius about the right half of the  $s$ -plane to a contour of the OLTF( $s$ ) magnitude and phase. Stability is determined by looking at the number of encirclements,  $N$ , of the point at  $(-1,$



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0). Using the argument principle for contour integrals, it can be shown that  $N=N_P-N_Z$  where  $N_P$  and  $N_Z$  are the number poles and zeros within the right half  $s$ -plane contour. The poles of the OLTF( $s$ ), which are known, are the zeros of the CLTF( $s$ ), as was shown in section 5.0 by expansion of the closed loop transfer function. The number of zeros,  $N_Z$  that are in the right half  $s$ -plane is known from the OLTF. The number of poles in the right half  $s$ -plane is then obtained as  $N_P=N-N_Z$ . If the point  $(-1, 0)$  is not encircled or intersected, then the closed loop system is stable. The intersection of the Nyquist contour with the negative real axis relative to the origin is the inverse of the gain margin. For example, if this distance is 'a' then the  $GM=1/a$ . The phase margin is obtained by drawing a line from the point of intersection of the Nyquist contour with a circle of radius one about the origin; to the origin and then calculating the angle relative to the negative real axis with positive being counter-clockwise. The Nyquist plot for the first example in section 5.1 describing Bode stability is shown in Figure 4. The top plot shows the contour is for a frequency range out to 100 Hz while the second two plots are expanded so that the phase margin and gain margin can be observed.



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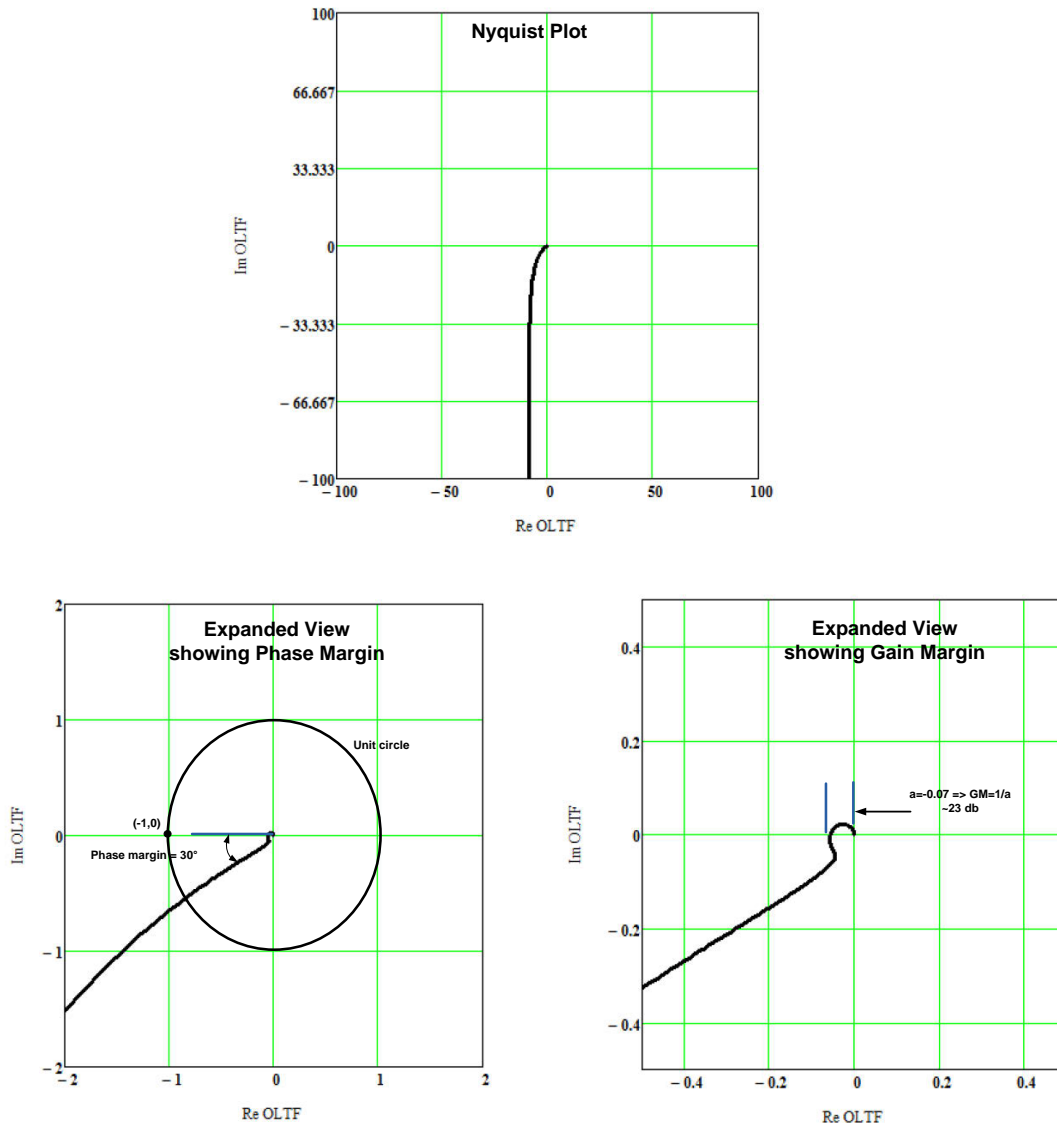


Figure 4 Nyquist Plots for Example 1

**6.0 Control Loop Performance Specifications:** The specification defines the desired feedback loop performance. From a practical perspective, the control loop designer must understand the user requirements to develop a satisfactory specification. Key characteristics often specified by a user are accuracy and response time for a defined command signal or range of signals and the disturbance and noise environment. From this information, the designer must develop the control requirements and specification. As discussed previously, accuracy and disturbance rejection are improved by a high gain controller, as is response time. As gain is effectively proportional to bandwidth, this implies a higher bandwidth improves performance. However as the bandwidth



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increase, the loop stability margins begin to decrease, more overshoot results, control elements can saturate, and noise between loop elements can be amplified. So the design is a tradeoff between the user requirements and control loop specifications that meet these requirements with sufficient stability margin. Response time can be specified in terms of rise time, overshoot, and settling time all of which are a function of bandwidth and the control compensator structure. In the frequency domain, the designer can specify the resonant peak, peak frequency, gain crossover frequency, bandwidth (which in some cases are directly related), and the minimum allowable phase and gain margins. The resonant peak is a maximum of the gain at the peak frequency. The other important specification is the control loop type which as discussed previously refers to the number of integrators within the loop; either due to the controller or the plant dynamics. The loop type or number of integrators will determine the accuracy with which the output follows an input command in the long term. This error is often referred to as the following error. The integrator also provides very high gain at low frequency which is good for disturbance rejection and improved accuracy; however each pole at  $s=0$  also provides a  $-90^\circ$  phase shift which can impact stability. The robustness of the controller design must also be considered. If the plant parameters change for whatever reason, what will happen to the loop stability margins? The controller design must account for these variations. This can be accomplished by designing for the worst case plant variation and ensuring the system meets the stability margins over the full range of plant variation; measuring the plant response (i.e. plant identification) and changing control parameters to account for the plant changes (either manually or on a scheduled basis), or a using an adaptive controller that automatically adjusts the control parameters as a function of the plant variations.

7.0 Controller Design: There are many control loop design techniques; most involve some form of loop shaping and/or algebraic pole-zero placement and cancellation. Loop shaping can be accomplished using Bode or Nyquist plot analysis. Visually, the Bode plot is easy to interpret and adjust the response especially if implemented in a mathematical computer aided design program (i.e. Matlab or MathCad). The Bode plot can be used with measured plant gain and phase response data, the plant-pole zero structure does not have to be known exactly (although the Bode design is still subject minimum phase criterion). Algebraic methods appear straightforward but require knowledge of the plant pole-zero structure and care must be taken to insure stable design. For example, pole and zero cancellation must be done with some sensitivity analysis because they are never known exactly and in some cases can change or drift. This is especially true if the poles and zeros are not stable; direct cancellation can result in an pole/zero pair that are both unstable (both roots in right half s-plane) as opposed cancellation of the unstable root. Common compensation elements used in a controller are proportional plus integral (PI) and/or derivative (PID) control, and first or second order lead and lag compensation as well





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as a cascade of these basic elements. The loop Type can be adjusted using the PI or PID controller. When using any control term with a pure integrator, the output should be limited or other measures taken to insure windup of the integrator does not occur. A couple of simple design examples are provided.

7.1 Frequency Domain Loop Shaping: Assume for a unity feedback loop the desired gain crossover frequency is 40 Hz and the phase margin  $> 55^\circ$ . This provides two design criteria to satisfy with the compensator:

- 1)  $OLG(f_{GC}) = 1$
- 2)  $PM = 180^\circ + \Phi(OLTF(f_{GC})) = 55^\circ$   
or  $\Phi(OLTF(f_{GC})) = -125^\circ$

From the definition of the open loop transfer function, this can be expressed as:

- 1)  $\left| \frac{C_N(j\omega_{GC})}{C_D(j\omega_{GC})} \right| \cdot \left| \frac{P_N(j\omega_{GC})}{P_D(j\omega_{GC})} \right| = 1 \quad ; \quad \omega_{GC} = 2 \cdot \pi \cdot f_{GC}$
- 2)  $\Phi(C(j\omega_{GC}) \cdot P(j\omega_{GC})) = \Phi(C_N(j\omega_{GC})) + \Phi(P_N(j\omega_{GC})) - \Phi(C_D(j\omega_{GC})) - \Phi(P_D(j\omega_{GC})) = -125^\circ$

The plant is given by:

$$P(s) = \frac{K_L}{s^2}$$

The plant parameter  $K_L$  is the scaling factor. For a double integrator, the phase lag is  $-180$  so it can be assumed that to obtain the desired phase response (i.e.  $> -180^\circ$ ) a lead compensator is required. With a lead compensator, the zero frequency is less than the pole frequency while for a lag compensator the zero frequency is greater than the pole frequency. The compensator will have the form:

$$C(s) = \frac{K_p \cdot 2 \cdot \pi \cdot 160 \cdot (s + \alpha)}{s + 2 \cdot \pi \cdot 160}$$

The pole frequency at 160 Hz is chosen as the roll-off frequency for the loop. It effectively limits the frequency response and reduces higher frequency noise. Representing the poles and zeros in terms of a frequency (i.e.  $2\pi f$ ) provides for better visualization of the design process and pole/zero location. It is also useful to cancel plant parameters with a scaling factor so effectively the loop response is shaped only by the normalized plant response and the compensator. The compensator is then expressed as:



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$$C(s) = \frac{K_{P0}}{K_L} \frac{\alpha \cdot \left(\frac{s}{2 \cdot \pi \cdot 160} + 1\right)}{\frac{s}{2 \cdot \pi \cdot 160} + 1} \quad \text{for} \quad K_P = \frac{K_{P0}}{K_L}$$

The OLTF is then:

$$OLTF(s) = C(s) \cdot P(s) = K_{P0} \frac{\alpha \cdot \left(\frac{s}{2 \cdot \pi \cdot 160} + 1\right)}{\left(\frac{s}{2 \cdot \pi \cdot 160} + 1\right)} \cdot \frac{1}{s^2}$$

This will be a Type 2 control loop. The values for the compensator gain and zero are obtained by first obtaining the value of the zero from the phase margin condition and solving for the gain from the OLG condition. From this expression, the phase components are given by:

$$\begin{aligned} \Phi(C_N(j\omega_{GC})) &= \text{atan}\left(\frac{\omega_{GC}}{\alpha}\right) & ; & \quad \Phi(C_D(j\omega_{GC})) = \text{atan}\left(\frac{\omega_{GC}}{2 \cdot \pi \cdot 160}\right) \\ \Phi(P_N(j\omega_{GC})) &= 0 & ; & \quad \Phi(P_D(j\omega_{GC})) = -180 \end{aligned}$$

Substituting into the expression for phase margin

$$-125^\circ = \text{atan}\left(\frac{\omega_{GC}}{\alpha}\right) - \text{atan}\left(\frac{\omega_{GC}}{2 \cdot \pi \cdot 160}\right) - 180$$

or

$$55^\circ = \text{atan}\left(\frac{2 \cdot \pi \cdot 40}{\alpha}\right) - \text{atan}\left(\frac{2 \cdot \pi \cdot 40}{2 \cdot \pi \cdot 160}\right) = \text{atan}\left(\frac{2 \cdot \pi \cdot 40}{\alpha}\right) - 14^\circ$$

Solving for  $\alpha$

$$69^\circ = \text{atan}\left(\frac{2 \cdot \pi \cdot 40}{\alpha}\right)$$

or

$$\alpha = \frac{2 \cdot \pi \cdot 40}{\tan(69^\circ)} = 97 = 2 \cdot \pi \cdot f_{ZC} \Rightarrow f_{ZC} = \frac{97}{2 \cdot \pi} = 15.4 \text{ Hz}$$

The phase margin criterion is satisfied with the compensator pole at 15.4 Hz. The magnitude criterion is now used to obtain the value of  $K_{P0}$  that satisfies the open loop gain condition at the gain crossover frequency.



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$$K_{P0} \cdot 2 \cdot \pi \cdot 160 \cdot \left| \frac{(j\omega_{GC} + 2 \cdot \pi \cdot 15.4)}{(j\omega_{GC} + 2 \cdot \pi \cdot 160)} \right| \cdot \left| \frac{1}{(j\omega_{GC})^2} \right| = 1$$

or

$$K_{P0} \cdot 2 \cdot \pi \cdot 160 \cdot \left| \frac{(j2 \cdot \pi \cdot 40 + 2 \cdot \pi \cdot 15.4)}{(j2 \cdot \pi \cdot 40 + 2 \cdot \pi \cdot 160)} \right| \cdot \left| \frac{1}{(j2 \cdot \pi \cdot 40)^2} \right| = K_{P0} \cdot 2 \cdot \pi \cdot 160 \cdot 0.26 \cdot \frac{1}{(2 \cdot \pi \cdot 40)^2} = 1$$

or

$$K_{P0} \cdot 2 \cdot \pi \cdot 41.6 \cdot \frac{1}{(2 \cdot \pi \cdot 40)^2} = 1 \Rightarrow K_{P0} = 2 \cdot \pi \cdot 38.5$$

The compensator gain that satisfies the open loop gain crossover frequency condition is then:

$$K_p = \frac{2 \cdot \pi \cdot 38.5}{K_L}$$

7.2 Compensator Derivation from a Desired CLTF(s): Another method for controller design is to specify a closed loop system model based upon a desired performance. For example, response time or bandwidth, overshoot, stability margins could be specified. With the closed loop transfer function defined and the plant transfer function known, the compensator can be determined. Assuming unity gain feedback and using the definition for the closed loop response:

$$CLTF(s) = \frac{P(s) \cdot C(s)}{1 + C(s) \cdot P(s)}$$

∴

$$C(s)P(s) = \frac{CLTF(s)}{1 - CLTF(s)}$$

or

$$C(s) = \frac{1}{P(s)} \cdot \frac{CLTF(s)}{1 - CLTF(s)} = \frac{P_D(s)}{P_N(s)} \cdot \frac{CLTF_N(s)}{CLTF_D(s) - CLTF_N(s)}$$

The compensator cancels out the plant and substitutes an open loop transfer function that gives the desired closed loop transfer function. There are some important constraints:

- If the plant has a pole in the right half s-plane (pole with a positive real part) then the desired CLTF(s) must be chosen such the CLTF<sub>D</sub>(s)-CLTF<sub>N</sub>(s) has the same root.
- If the plant has a zero in the right half s-plane then the desired CLTF(s) must have the same right half s-plane zero



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- To be realizable, the excess poles of the desired CLTF(s) must be equal or greater than the excess poles of the plant. If the numerator polynomial of a transfer function is of order  $n_z$  and the denominator polynomial of order  $n_p$  then the excess poles are defined as  $n_p - n_z$ .

A very simple example is provided with a desired CLTF(s) and plant of the form:

$$CLTF(s) = \frac{\frac{a}{b}s + 1}{\frac{s^2}{b} + \frac{a}{b} \cdot s + 1} \quad ; \quad P(s) = \frac{K_L}{s}$$
$$\therefore C(s) = \frac{s}{K_L} \frac{\frac{a}{b} \cdot s + 1}{\frac{s^2}{b} + \frac{a}{b} \cdot s + 1 - (\frac{a}{b} \cdot s + 1)}$$
$$C(s) = \frac{a}{K_L} \cdot \frac{s + \frac{b}{a}}{s}$$

The constants a, b define the desired response of the CLTF(s). Using this procedure basically results in a proportional plus integral (PI) controller to realize the form of a desired CLTF(s). There were no right half plane poles or zeros in this example and the excess pole criterion is also satisfied. Unfortunately, a real design is seldom this simple. To satisfy the excess pole criterion, it may be required to add more poles to the desired CLTF(s) than required to just obtain the desired performance. In this case, the CLTF(s) is constructed as the cascade of a transfer function that dominates the response and defines performance, and a transfer function with poles significantly beyond the desired bandwidth that will satisfy the excess pole constraint.

**8.0 Home Heating System:** The first example is a simplified version of the feedback control for heating a house. The physical configuration is illustrated in Figure 5. The house is idealized as a box filled with air at a uniform temperature  $T_C$ . The walls of the house are considered as pure resistance to heat transfer with no energy storage capacity. The overall coefficient of heat transfer is U and the heat transfer area is A. The outdoor environment temperature is  $T_e$  and varies with time thereby acting as a disturbance to the control system. The temperature  $T_C$  is measured by a temperature sensor in a thermostat or temperature controller mounted inside the house. The desired temperature can also be set by this device, or in today's environment may even interface to a smart phone with an APP that lets it be set remotely. It is assumed that the temperature is converted to a voltage (or current) with a scaling constant  $K_{TV}$  (with units Volt/ $^{\circ}$ F) and processed by electronics, or more likely in today's environment a micro-controller.



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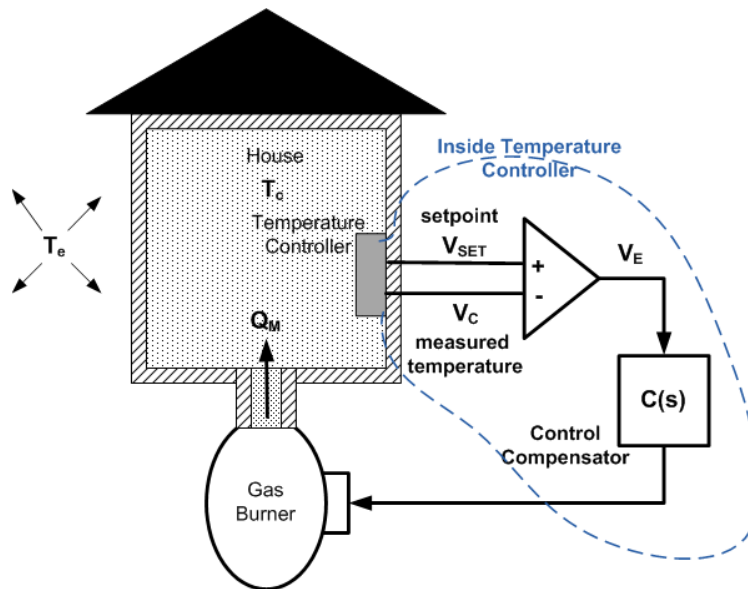


Figure 5 Concept Feedback Control Configuration for Heating a House

The controller takes the difference between the desired and actual temperature and generates a control signal to the furnace. This signal controls an actuator that increases or decreases the gas flow and effectively the amount of heat into the room. For this simple example, the actuation and heat generation are modeled as a linear process; in reality it is a complex process. In developing a model of the system it is assumed that initially the house is at equilibrium with constant values of  $T_C$  and  $T_e$ . The furnace will then be supplying sufficient heat to balance the heat loss to the environment. Any disturbance or change in the desired temperature will result in an increase or decrease in heat input from this original value. All variables should be considered deviations from their initial equilibrium condition. The simple first order thermal dynamic model for the heat balance in the house (the control loop plant) in the time domain is:

thermal energy stored = thermal energy in – thermal energy leaving the house

or 
$$M \cdot C_p \cdot \dot{T}_C(t) = Q_M(t) - U \cdot A \cdot (T_C(t) - T_e(t))$$

where  $Q_M$  – thermal energy in  $\frac{\text{BTU}}{\text{hr}}$

$M$  – mass of air in house

$C_p$  – specific heat of air at constant pressure

assume  $U \cdot A = 150 \frac{\text{BTU}}{\text{Hr} \cdot ^\circ\text{F}}$  ;  $M \cdot C_p = 180 \frac{\text{BTU}}{^\circ\text{F}}$

Converting to the frequency domain and rearranging terms:



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$$T_C(s) = \frac{Q_M(s)}{U \cdot A \cdot (\tau \cdot s + 1)} + \frac{T_e(s)}{\tau \cdot s + 1}$$

where  $\tau = \frac{M \cdot C_P}{U \cdot A} = 1.2 \text{ Hr} = 4320 \text{ sec}$

This represents the model of the plant; in terms of previous notation:

$$P(s) = \frac{1}{U \cdot A \cdot (\tau \cdot s + 1)} = \frac{1}{150 \cdot (4320 \cdot s + 1)} \frac{\text{°F}}{\text{BTU} / \text{Hr}}$$

A feedback control system measures the house temperature  $T_C$  and compares it to a desired temperature  $T_S$  or the set point. The control loop is shown in Figure 6 with the yet undefined controller  $C(s)$ .

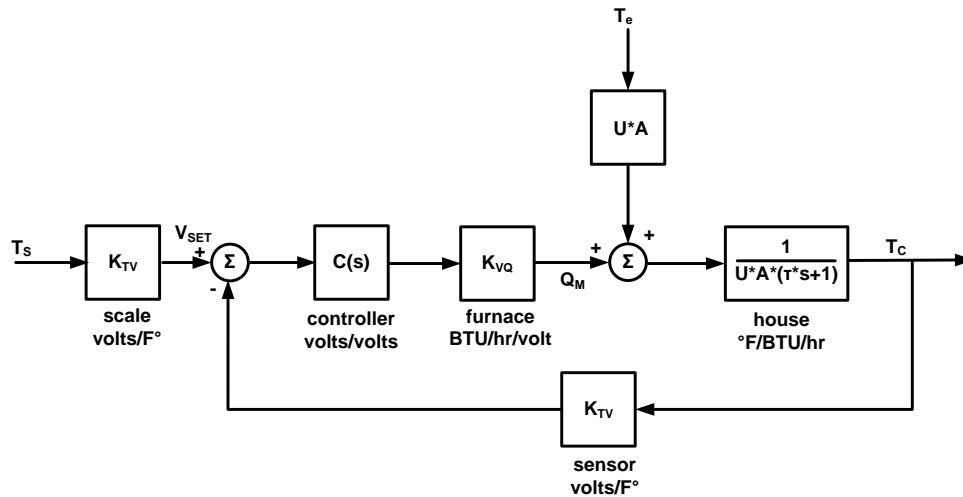


Figure 6 Block diagram of home heating control servo loop

From the block diagram, the dynamics of the loop can be expressed as:

$$T_C(s) = P(s) \cdot [U \cdot A \cdot T_e(s) + C(s) \cdot (K_{TV} \cdot (T_S(s) - T_C(s)))]$$

assume  $K_{TV} = 0.067 \frac{\text{Volt}}{\text{°F}}$

Solving for  $T_C$ :

$$T_C(s) = \frac{P(s) \cdot U \cdot A \cdot T_e(s)}{1 + K_{TV} \cdot P(s) \cdot C(s)} + \frac{P(s) \cdot C(s) \cdot K_{TV} \cdot T_S(s)}{1 + K_{TV} \cdot P(s) \cdot C(s)}$$



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The error between the set point temperature and the actual temperature is given by:

$$T_S(s) - T_C(s) = -\frac{P(s) \cdot U \cdot A \cdot T_e(s)}{1 + K_{TV} \cdot P(s) \cdot C(s)} + \frac{T_S(s)}{1 + K_{TV} \cdot P(s) \cdot C(s)}$$

As discussed previously, the error is a function of the magnitude of the outside and set point temperatures and the attenuation provided by the controller gain. The simplest controller is a proportional gain or  $C(s) = K_P$  with units of BTU/Hr/Volt. Substituting into the previous expression and using the plant definition:

$$\begin{aligned} T_S(s) - T_C(s) &= \frac{U \cdot A \cdot T_e(s)}{U \cdot A \cdot (\tau \cdot s + 1) + K_{TV} \cdot K_P} + \frac{U \cdot A \cdot (\tau \cdot s + 1) \cdot T_S(s)}{U \cdot A \cdot (\tau \cdot s + 1) + K_{TV} \cdot K_P} \\ &= \lambda_s \cdot \left[ \frac{T_e(s)}{\lambda_s \cdot \tau \cdot s + 1} + \frac{(\tau \cdot s + 1)T_S(s)}{\lambda_s \cdot \tau \cdot s + 1} \right] \end{aligned}$$

$$\text{where } \lambda_s = \frac{U \cdot A}{U \cdot A + K_{TV} \cdot K_P}$$

Stability is not an issue, as the open loop has only one real negative pole. The term  $\lambda_s$  is basically the attenuation factor whose magnitude is based upon the value of  $K_P$ . The time constant of the system will also be reduced by this factor. As this is a Type 0 system, there will be a steady state error to a step change. For a step change in  $T_e$  and  $T_S$ , the steady state error is given by:

$$E_{SS} = \lim_{s \rightarrow 0} s \cdot (T_S(s) - T_C(s)) = \lim_{s \rightarrow 0} s \cdot \left[ \frac{\lambda_s}{\lambda_s \cdot \tau \cdot s + 1} \cdot \frac{\delta T_e}{s} + \frac{\lambda_s \cdot (\tau \cdot s + 1)}{\lambda_s \cdot \tau \cdot s + 1} \cdot \frac{\delta T_S}{s} \right]$$

$$E_{SS} = \lambda_s \cdot (\delta T_e + \delta T_S)$$

The step change in the input will most likely come from a new set point, as the temperature of the environment would not be expected to change instantaneously so ( $\delta T_e=0$ ):

$$E_{SS} = \lambda_s \cdot \delta T_S \quad \text{step change}$$

The gain  $K_P$  is chosen to provide an attenuation factor  $\lambda$  that is consistent with the amount of allowable offset that can be tolerated between the new set point and actual temperature. For example with the gain was chosen as:

$$K_P = \frac{100 \cdot U \cdot A}{K_{TV}}$$



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The resulting attenuation would be a factor of 100 or -40 dB. The achievable gain will also depend upon the actual drive characteristics of the furnace. The gain crossover can be determined by a Bode plot or estimated directly from the OLTf as  $f_{GC} \sim 0.004$  Hz.

With a proportional plus integral controller (PI) the system is Type 1 and the steady state offset to a step would be zero. The PI compensator has the form:

$$C(s) = K_p \left( \frac{s + \alpha}{s} \right)$$

$$K_p = \frac{0.095 \cdot U \cdot A \cdot \tau}{K_{TV}}$$

$$\alpha = 0.045$$

The open loop transfer function is now:

$$K_{TV} \cdot P(s) \cdot C(s) = K_{TV} \left( \frac{1}{U \cdot A \cdot (\tau \cdot s + 1)} \right) K_p \left( \frac{s + \alpha}{s} \right)$$

Closed loop system denominator is now a quadratic polynomial. Since all coefficients are positive, the roots will have negative real parts so that the system will be stable. The stability margins are obtained using a Bode plot. The open loop gain and phase are shown in Figure 7.

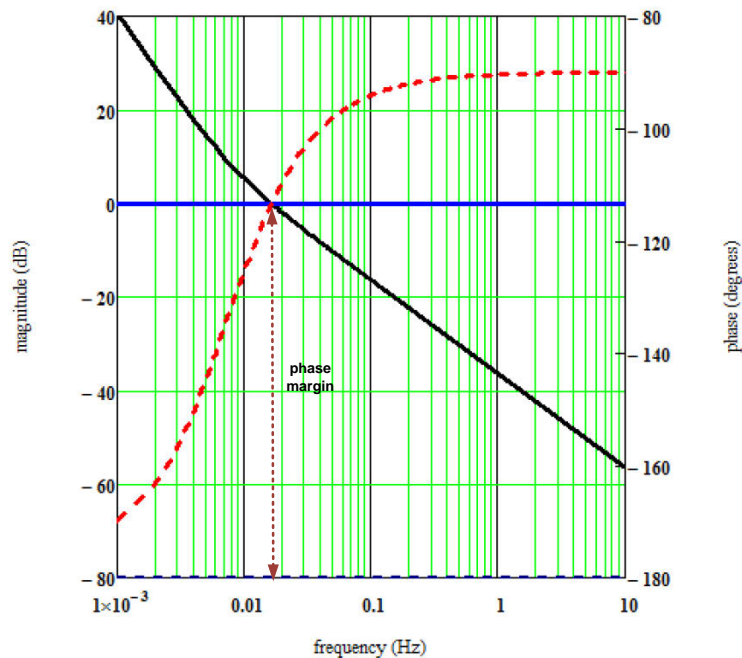


Figure 7 Bode plot for Example with PI controller





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The gain crossover frequency is ~0.0167 Hz, the PM~64° and the gain margin infinite since the PI integrator lag is offset by the zero so the net loop lag is only the lag of the plant which being first order is only 90°. Care must be taken in an actual design to ensure the higher bandwidth does not put too much demand on the furnace drive. The expression for the error for this compensator is:

$$T_S(s) - T_C(s) = \frac{s \cdot U \cdot A \cdot T_e(s)}{U \cdot A \cdot s \cdot (\tau \cdot s + 1) + K_{TV} \cdot K_P \cdot (s + \alpha)} + \frac{s \cdot U \cdot A \cdot (\tau \cdot s + 1) \cdot T_S(s)}{U \cdot A \cdot s \cdot (\tau \cdot s + 1) + K_{TV} \cdot K_P \cdot (s + \alpha)}$$

$$= \lambda_r \cdot s \cdot \left[ \frac{T_e(s)}{b_2 \cdot s^2 + b_1 \cdot s + 1} + \frac{(\tau \cdot s + 1)T_S(s)}{b_2 \cdot s + b_1 \cdot s + 1} \right]$$

where  $\lambda_r = \frac{U \cdot A}{K_{TV} \cdot K_P \cdot \alpha}$  ;  $b_2 = \frac{U \cdot A \cdot \tau}{K_{TV} \cdot K_P \cdot \alpha}$  ;  $b_1 = \frac{U \cdot A + K_{TV} \cdot K_P}{K_{TV} \cdot K_P \cdot \alpha}$

For a step change, the steady state error will be zero since:

$$E_{SS} = \lim_{s \rightarrow 0} s \cdot (T_S(s) - T_C(s)) = \lim_{s \rightarrow 0} \left[ \frac{\lambda_r \cdot s^2}{b_2 \cdot s^2 + b_1 \cdot s + 1} \cdot \left[ \frac{\delta T_e}{s} + (\tau \cdot s + 1) \cdot \frac{\delta T_S}{s} \right] \right] = 0$$

For a ramp or steadily increasing temperature change, the steady state error is:

$$E_{SS} = \lim_{s \rightarrow 0} s \cdot (T_S(s) - T_C(s)) = \lim_{s \rightarrow 0} \left[ \frac{\lambda_r \cdot s^2}{b_2 \cdot s^2 + b_1 \cdot s + 1} \cdot \left[ \frac{\delta \dot{T}_e}{s^2} + (\tau \cdot s + 1) \cdot \frac{\delta \dot{T}_S}{s^2} \right] \right]$$

$$E_{SS} = \lambda_r \cdot (\delta \dot{T}_e + \delta \dot{T}_S)$$

The set point will normally not change as a ramp, however the temperature of the environment can so that most likely the steady state error for this case would be ( $\delta(dT_S/dt) = 0$ ):

$$E_{SS} = \lambda_r \cdot \delta \dot{T}_e \quad \text{ramp}$$

Using the definition of  $K_P$  for this case,  $\lambda_r=0.054$  or -25 dB with the PI controller.

**9.0 Motion Control Feedback Systems:** Position and rate feedback control systems are used in many applications; cameras for surveillance and in the movie industry, communication and radar antennas, positioning concentrating solar collectors, computer disk control, pointing lasers, etc. A typical pan tilt zoom camera assembly<sup>(1)</sup> is shown in Figure 8. For this application, the plant or load (camera as shown in the figure) is mounted on a rotary stage driven by a motor. A servo amplifier converts control commands to a motor drive signal, for example a control voltage to a motor drive current which is applied to the motor stator windings to generate a torque causing the rotary stage on which the load is mounted to rotate. The basic elements of the control loop are shown in Figure 9.



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1. [http://commons.wikimedia.org/wiki/File:Axis\\_214\\_PTZ\\_Camera.jpg](http://commons.wikimedia.org/wiki/File:Axis_214_PTZ_Camera.jpg)

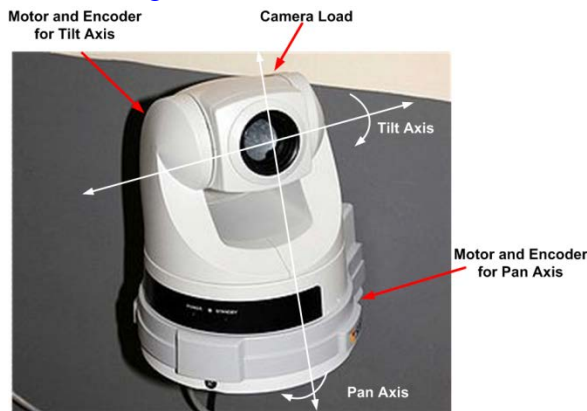


Figure 8 AXIS Pan Tilt Zoom Camera Assembly

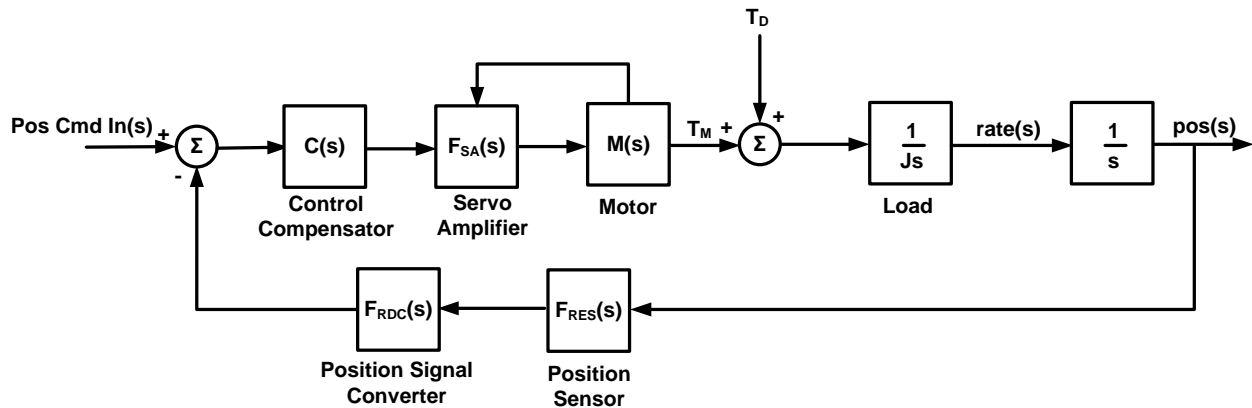


Figure 9 Basic Block Diagram for a Motion Control Feedback Loop

The position command is compared to the measured position, calculating the error between the two positions. The control compensator uses the error to calculate a drive command to the motor which is applied through the servo amplifier. The output of the motor is a torque that is applied to the load along with any disturbances. The motor feedback to the servo amplifier is a back electro-motive force (EMF) which will be discussed later. As shown, the load is simply modeled as pure inertia and neglects the effects of friction which in general depend on the rate signal. The



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position feedback sensor and signal converter work in tandem to generate a measured signal in the units used by the servo loop; normally the net feedback gain is unity.

9.1 Motor and Servo Amplifier Models: Many motors used for position control applications are DC servo motors, although AC drives can also be used. The motor assumed for the example is a DC servo motor. A simple model for a DC motor includes three equations:

Motor voltage to current:

$$V_M(t) = R \cdot I(t) + L \cdot \dot{I}(t) + K_e \cdot \text{rate}(t)$$

where: R – winding resistance ; I(t) – motor current

L – winding inductance ; rate(t) – motor rotor angular rate

$K_e$  – back EMF voltage constant

Motor current to motor torque ( $T_M(t)$ ):

$$T_M(t) = K_T \cdot I(t)$$

where:  $K_T$  – motor torque constant

Motor torque to load torque:

$$J \cdot \text{rate}'(t) + B \cdot \text{rate}(t) = T_M(t) + T_D(t)$$

where: J – motor plus load inertia

B – load viscous damping constant

The motor model is shown as a block diagram in Figure 10.

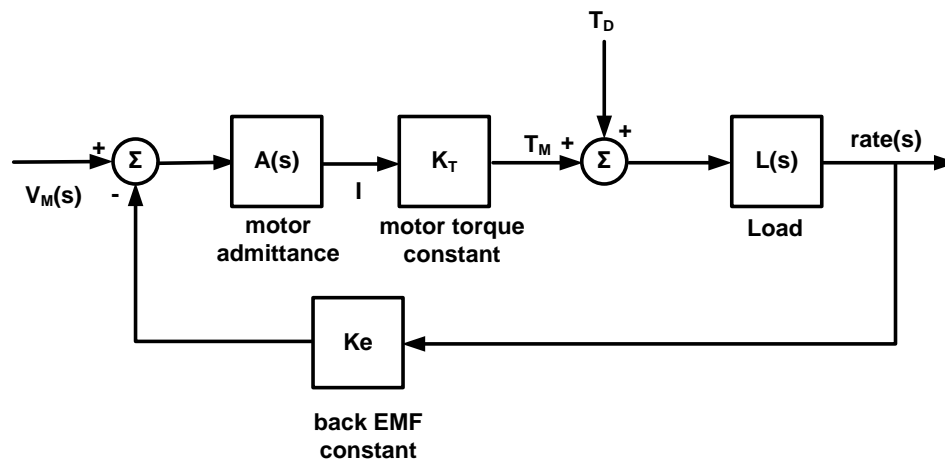


Figure 10 DC Motor Block Diagram



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The transfer function from motor voltage and disturbance torque in to the rate out can be expressed as:

$$rate(s) = \frac{K_T \cdot A(s) \cdot L(s)}{1 + K_e \cdot K_T \cdot A(s) \cdot L(s)} \cdot V_M(s) + \frac{L(s)}{1 + K_e \cdot K_T \cdot A(s) \cdot L(s)} \cdot T_D(s)$$

where:  $L(s) = \frac{1}{J \cdot s + B}$  load response

$$A(s) = \frac{1}{L \cdot s + R} \text{ winding admittance}$$

The electromechanical and electrical time constants, with viscous damping assumed negligible ( $B \sim 0$ ), can be derived approximately from this expression as:

$$\tau_M = \frac{R \cdot J}{K_e \cdot K_T} \quad ; \quad \tau_E = \frac{L}{R}$$

There are two types of servo amplifiers used to drive the motors; voltage and current. With a voltage servo amplifier the motor voltage is related to the command voltage by the scaling factor or gain  $K_V$  as:

$$V_M = K_V \cdot V_C$$

The model of the voltage servo amplifier and motor from a command voltage and disturbance in to the rate out can then be expressed as:

$$rate(s) = \frac{K_T \cdot A(s) \cdot L(s) \cdot K_V}{1 + K_e \cdot K_T \cdot A(s) \cdot L(s)} \cdot V_C(s) + \frac{L(s)}{1 + K_e \cdot K_T \cdot A(s) \cdot L(s)} \cdot T_D(s)$$

The block diagram for the current feedback servo amplifier configuration is shown in Figure 11. The current feedback loop is a high gain servo loop that controls the motor current. The amplifier  $G_I(s)$  has very high gain and  $K_A$  is effectively the scaling constant (Amps/Volt) of the current amplifier.



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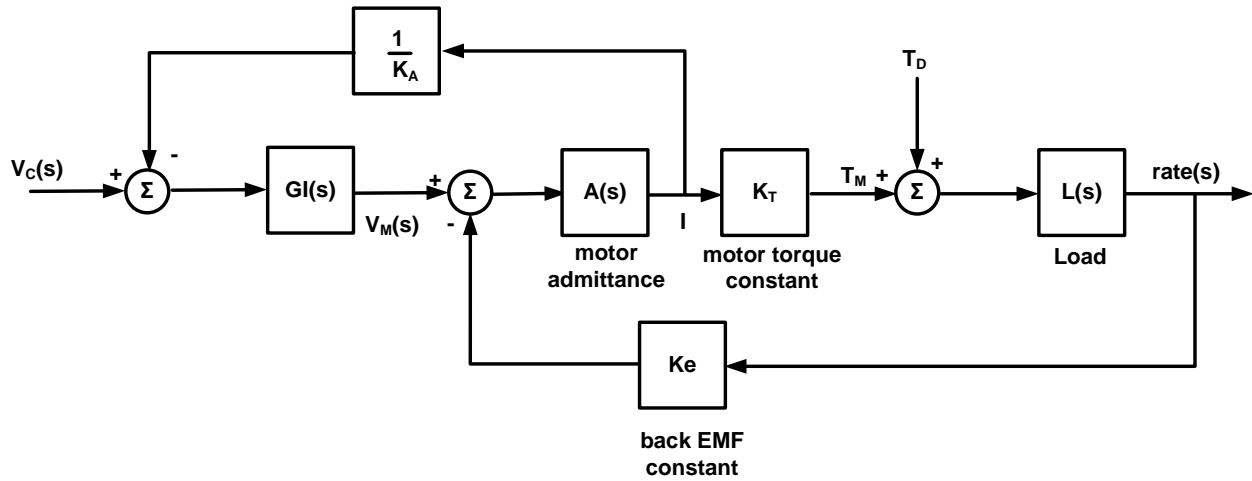


Figure 11 Motor Block Diagram with Current Feedback Servo Amplifier with Scaling  $K_A$

The transfer function between a command voltage and disturbance in and the rate out is given by:

$$rate(s) = \frac{K_T \cdot A(s) \cdot L(s) \cdot GI(s)}{1 + K_T \cdot A(s) \cdot L(s) \cdot \left( K_e + \frac{GI(s)}{K_A \cdot K_T \cdot L(s)} \right)} \cdot V_C(s) + \frac{L(s) \cdot \left( 1 + \frac{A(s) \cdot GI(s)}{K_A} \right)}{1 + K_T \cdot A(s) \cdot L(s) \cdot \left( K_e + \frac{GI(s)}{K_A \cdot K_T \cdot L(s)} \right)} \cdot T_D(s)$$

The gain of the current amplifier,  $GI(s)$ , is chosen such that within loop bandwidth:

$$\left| \frac{GI(s)}{K_A \cdot K_T \cdot L(s)} \right| \gg K_e$$

For this condition, the transfer function reduces to:

$$rate(s) = \frac{K_T \cdot A(s) \cdot L(s) \cdot GI(s)}{1 + A(s) \cdot \frac{GI(s)}{K_A}} \cdot V_C(s) + L(s) \cdot T_D(s)$$

Again as  $GI(s)$  is a high gain amplifier within the loop bandwidth:

$$\left| \frac{GI(s)}{K_A \cdot A(s)} \right| \gg 1$$

The final expression can then be written as:

$$rate(s) = L(s) \cdot K_T \cdot K_A \cdot V_C(s) + L(s) \cdot T_D(s) = L(s) \cdot (K_T \cdot K_A \cdot V_C(s) + T_D(s))$$



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With a current feedback servo amplifier, the effect of the back EMF and motor admittance is nearly eliminated and the relationship expressed by the servo amplifier scaling constant  $K_A$ .

9.2 Position and Velocity Feedback Sensors: Typical rotary position feedback sensors are resolvers and encoders. For rotary applications, a feature of both these devices is that they can be mounted directly on the drive shaft and are capable of providing a full  $360^\circ$  continuous measurement range. There are several units of measure that can be used when describing angular position, degrees, radians, arc-minutes or arc-seconds. There are 60 arc-minutes per degree and  $\pi/180$  radians per degree which equates to  $\pi/(180*60)$  radians per arc-minute. There are 60 arc-seconds per arc-minute or therefore  $\pi/(180*3600)$  radians per arc-second ( $4.85*10^{-6}$ ). As degrees are a familiar unit of measure, this will be used in all examples, however actual application often used these other units. Resolvers provide absolute angular position with electrical errors ranging from  $0.1^\circ$  to as low as 10-20 arc-seconds. Accuracy depends upon size and angular range. Resolvers consist of a stator and rotor, similar to a motor. This construction makes them very robust in the presence of shock and vibration disturbances and reduces their sensitivity to temperature variations when compared to encoders. The resolver stator consists of two windings positioned at right angles to each-other. The rotor has a third winding that is energized with a sinusoidal signal and rotates relative to the stator. The signal in the rotor winding induces a signal in both stator windings whose magnitude varies as a function of the rotation angle. The voltage induced in one stator winding is in quadrature to the voltage in the other winding. Quadrature refers to the  $90^\circ$  phase relationship between the signals of the two windings. The output ratio of the two stator windings signals is proportional to the absolute angular position (arc-tangent of the ratio). An electronic circuit, the resolver to digital converter (RDC), is normally used to convert the conditioned resolver signal to a digital output that can be read by a computer.

Encoders are fabricated in both optical and magnetic configurations. The most basic encoder contains a rotating disk that interrupts transmission between a transmitter and a receiver. With an optical encoder, the optical source is typically a Light Emitting Diode (LED) and the receiver a photo detector. The disk has coded patterns of transparent and opaque sectors that interrupt the light measured by the photo detector. The number of counts (pulses) obtained as the disk rotates is a measure of the angular position. With a magnetic encoder, the LED is replaced with a magnet, the photo detector by a magnetic pick-up sensing element, and the disk is replaced by a rotating disk, similar to a gear, made of ferrous metal. As the gear rotates, the teeth disturb the magnetic flux emitted by the permanent magnet causing the flux field to expand and collapse.



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These changes are again sensed as pulses by the magnetic pick-up detector. The simplest and least expensive encoders are incremental. These devices do not provide an absolute measure of angular position as with the resolver. They have to be 'homed' periodically to obtain an estimate of absolute position and will lose the position reference upon power reset. The most common type of incremental encoder has two output channels that sense position. Two code tracks, referred to as channel A and channel B, have sectors positioned 90 deg out of phase that provide quadrature signals that can be used to detect position and direction of rotation. Each channel provides N counts per revolution (usually a square wave TTL level). Shifting the signals from the two channels by a quarter of a cycle enables the direction of rotation to be determined based upon which channel is leading the other. The shift also increases resolution as each encoder cycle can be divided into four quarters each called a quadrature count. An encoder with N cycles per revolution produces 4N quadrature counts per revolution. Resolution less than an arc-second can be obtained with interpolation electronics/software. Optical encoders normally use a glass substrate for the rotating disk, which makes them less desirable for high disturbance environments.

There also are absolute encoders which are normally larger and have more complex disk patterns containing 4-6 tracks. They generate a unique word (i.e. BCD or gray code) for every angular position of the shaft. Magnetic encoders also come in both absolute and incremental versions. They normally have less resolution than optical encoders but are more robust in a high disturbance environment. Other types of position sensors sometimes used for rotary angular position measurement include potentiometers, rotary and linear variable differential transformers. However these devices normally provide only a limited angle measurement range as opposed to the resolver or encoder. They do have one advantage in that they can generate a voltage directly proportional to angle requiring only a scale factor for conversion.

Velocity or angular rate can be measured or derived from position. Many resolver RDC cards as well as encoder conversion circuits can provide an estimate of velocity. Similarly it can be determined by differentiation of the position signal. If performed digitally at a high sample rate relative to the bandwidth of the rate signal; a good rate estimate can often be generated. A tachometer can also be used to measure rate. The analog tachometer resembles a small motor however the gauge of the wire is very fine. It is effectively a motor in reverse. As the shaft is turned by a motor it generates a voltage proportional to the angular rate requiring only a scale factor for conversion.

All the devices described have an associated frequency dependent bandwidth which is a function of the sensor as well as the conversion process. In a detailed design this must be accounted for. If the bandwidth of the feedback sensor is high relative to the bandwidth of the feedback control loop, it will have minimal impact on the response. If not, lead compensation may be required. As discussed, each device has a unique method of measuring angle and the actual output may be



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volts, counts, etc. However the feedback path is scaled such that it provides a signal in the same units as the command input and is therefore a unity feedback path. In the examples that follow, the position feedback sensor is assumed to be an ideal sensor with effectively infinite bandwidth and scaled to unity feedback gain.

9.3 Motion Control Feedback Loop Analysis: A position feedback control loop is shown in Figure 12. It uses a current feedback servo amplifier.

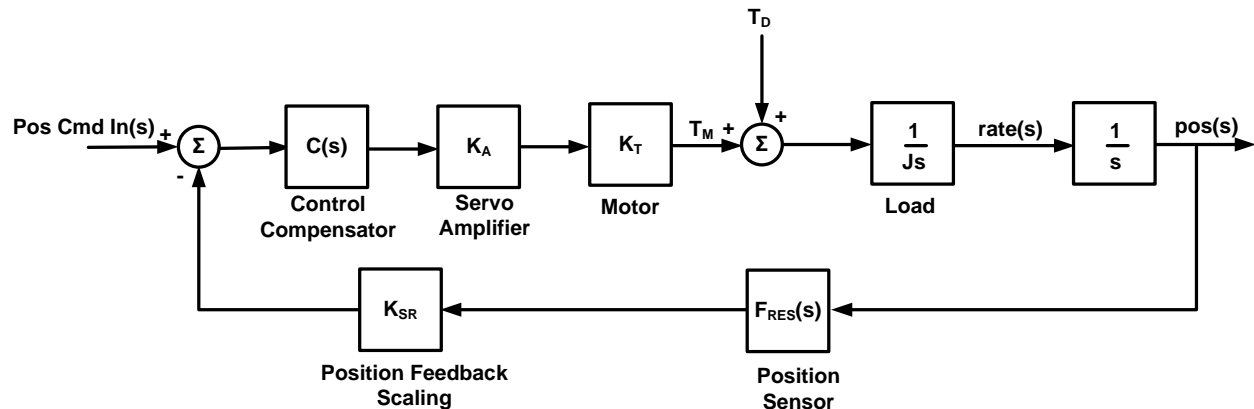


Figure 12 Block Diagram for a Motion Control Feedback Loop with Current Feedback Servo Amp

The constants and control compensator transfer function are given as:

- $K_T=0.1$  in-lb/amp
- $K_A=0.5$  amp/volt
- $C(s)=2\pi*40*K_S*2\pi*160*(s+2\pi*16)/(s+2\pi*160)$  volt/deg
- $L(s)=1/(J*s^2)$  ;  $J=0.0001$  in-lb-sec<sup>2</sup>
- $K_{SR} * F_{RES}(s) = 1$  (deg/deg) per assumption of ideal sensor
- $K_S=J/(K_A*K_T)$

This is a lead compensator with a zero at 16 Hz to shape the loop and a roll off pole at 160 Hz for noise suppression. There are already two integrators in the loop due to the physical dynamics of converting acceleration to position; which also means this is a Type 2 servo loop. There are two scaling constants, one in the forward path and the other in the feedback path. The scaling constant in the feedback path provides for unity gain feedback or  $K_{SR}*F_{RES}=1$ . The scaling constant in the forward path cancels the motor drive and plant scaling parameters and allows the





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control compensator to effectively set the loop bandwidth in the analysis. The open loop transfer function is given by:

$$OLTF(s) = \frac{K_A \cdot K_T}{J \cdot s^2} \cdot \frac{K_A \cdot K_T}{J} \cdot 2 \cdot \pi \cdot 40 \cdot 2 \cdot \pi \cdot 160 \cdot (s + 2 \cdot \pi \cdot 16)$$
$$= \frac{2 \cdot \pi \cdot 40 \cdot (s + 2 \cdot \pi \cdot 16)}{s^2} \cdot \frac{2 \cdot \pi \cdot 160}{s + 2 \cdot \pi \cdot 160}$$

It can be seen from the Bode plot in Figure 13 that the gain cross over frequency is at ~40 Hz. The phase margin is about 54° with an infinite gain margin. Closed loop response shown solid green line, with unity gain to ~40 Hz and some minor closed loop peaking. The -3 dB BW is 60 Hz.

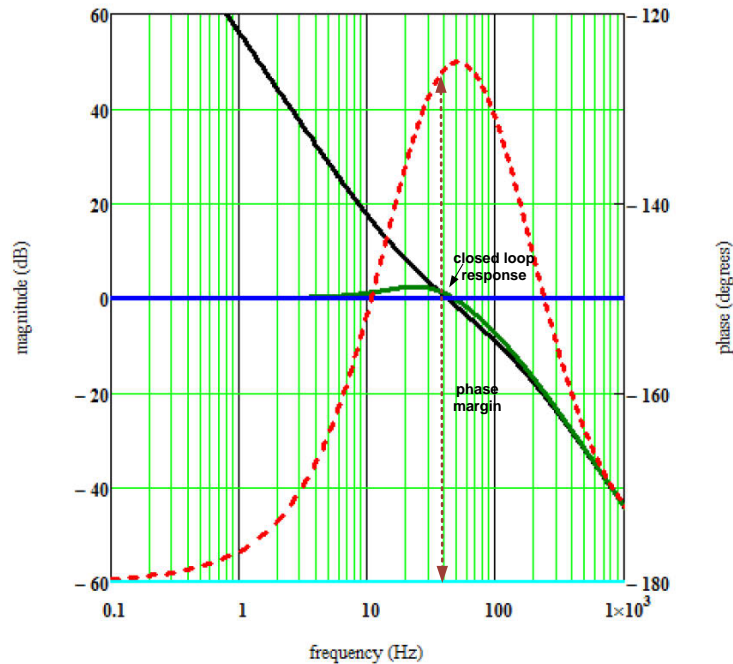


Figure 13 Bode plot for Motion Control Feedback Loop Example 1

The compliance plot of a disturbance acceleration (or torque as  $A=T/J$ ) input to the control loop acceleration output is shown in Figure 14 and effectively provides -16 dB of rejection at 10 Hz.



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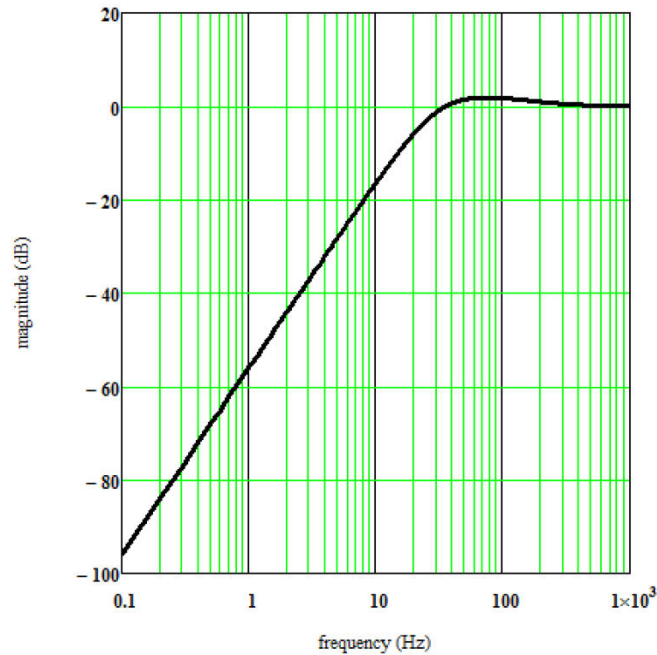


Figure 14 Torque Compliance Plot for Motion Control Example 1

The same system is analyzed, however this time a velocity or rate feedback loop is included in the design. The rate loop is the inner loop while the actual positioning loop is the outer loop; as shown in Figure 15.

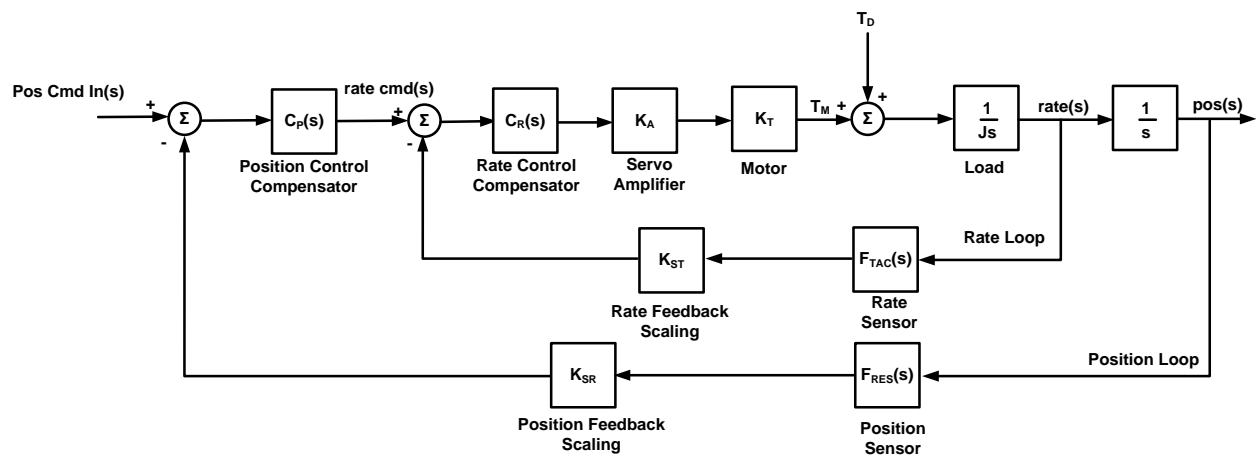


Figure 15 Motion Control with Position and Rate Feedback

The constants are the same as for the previous design however the control compensator transfer functions have been modified as (dps=degrees per second):

- $K_T=0.1$  in-lb/amp



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- $K_A=0.5$  amp/volt
- $C_R(s)=2\pi \cdot 100 \cdot K_S \cdot 2\pi \cdot 400 \cdot (s+2\pi \cdot 35)/[s \cdot (s+2\pi \cdot 400)]$  volt/dps
- $K_S=J/(K_A \cdot K_T)$
- $C_P(s)=2\pi \cdot 20 \cdot 2\pi \cdot 160 \cdot (s+2\pi \cdot 10)/[s \cdot (s+2\pi \cdot 160)]$  volt/dps
- $L(s)=1/(J \cdot s^2)$  ;  $J=0.0001$  in-lb-sec<sup>2</sup>
- $K_{ST} \cdot F_{TAC}(s) = 1$  (dps/dps) per assumption of ideal sensor
- $K_{SR} \cdot F_{RES}(s) = 1$  (deg/deg) per assumption of ideal sensor
- 

Both compensators now have the form of proportional plus integral compensators cascaded with a low pass roll off filter. The transfer function for the inner rate loop from the rate command input to the rate output is given by:

$$\begin{aligned} rate(s) &= \frac{\frac{K_A \cdot K_T}{J \cdot s} \cdot C_R(s)}{1 + \frac{K_A \cdot K_T}{J \cdot s} C_R(s)} \cdot rate\ cmd(s) + \frac{1}{J \cdot s} \cdot \frac{1}{1 + \frac{K_A \cdot K_T}{J \cdot s} C_R(s)} \cdot T_D(s) \\ &= RCLTF(s) \cdot rate\ cmd(s) + RCOMP(s) \cdot T_D(s) \end{aligned}$$

where

$$RCLTF(s) = \frac{\frac{K_A \cdot K_T}{J \cdot s} \cdot C_R(s)}{1 + \frac{K_A \cdot K_T}{J \cdot s} C_R(s)} \quad ; \quad RCOMP(s) = \frac{1}{J \cdot s} \cdot \frac{1}{1 + \frac{K_A \cdot K_T}{J \cdot s} C_R(s)}$$

The rate open loop transfer function is given by:

$$\begin{aligned} ROLTF(s) &= \frac{K_A \cdot K_T}{J \cdot s} \cdot C_R(s) \\ &= \frac{K_A \cdot K_T}{J \cdot s} \cdot \frac{K_S \cdot 2 \cdot \pi \cdot 100 \cdot 2 \cdot \pi \cdot 400 \cdot (s + 2 \cdot \pi \cdot 35)}{s \cdot (s + 2 \cdot \pi \cdot 400)} \\ &= \frac{2 \cdot \pi \cdot 100 \cdot 2 \cdot \pi \cdot 400 \cdot (s + 2 \cdot \pi \cdot 35)}{s^2 \cdot (s + 2 \cdot \pi \cdot 400)} \end{aligned}$$

A Bode plot of the rate open loop gain and phase is shown in Figure 16.



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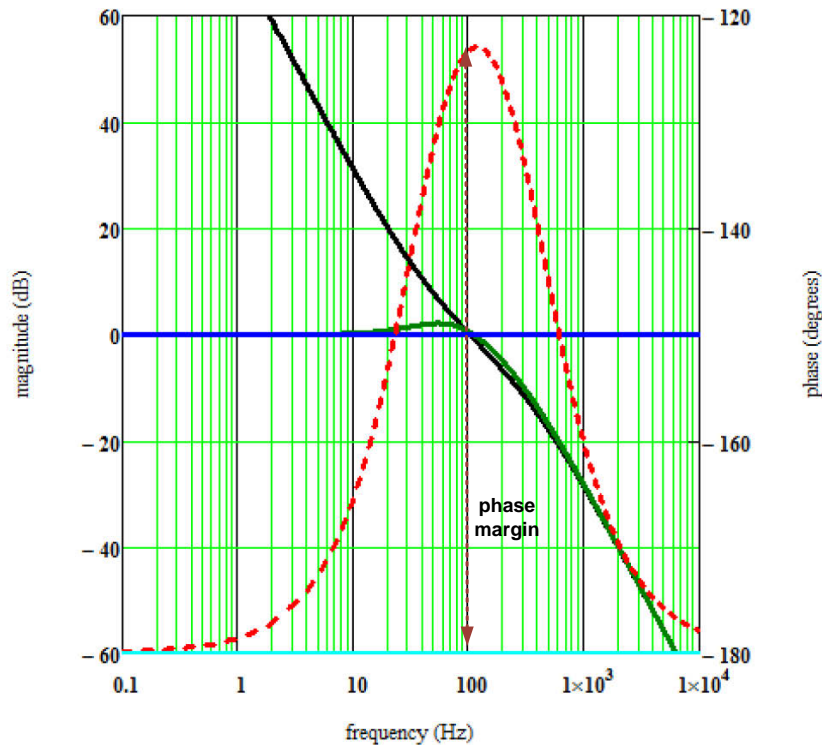


Figure 16 Bode Plot for Example 2 Inner Rate Loop

The gain crossover frequency is 100 Hz, the PM=56.8° and the GM is infinite; effectively this should be a well behaved loop with a fast response. The closed loop response is shown in green. The outer position loop can be derived from the rate loop noting that  $\text{rate}(s) = s \cdot \text{pos}(s)$  and  $\text{rate cmd} = C_P(s) \cdot (\text{pos cmd in}(s) - \text{pos}(s))$ . Using these relationships the expression for the position loop can be expressed as:

$$s \cdot \text{pos}(s) = RCLTF(s) \cdot C_P(s) (\text{Pos Cmd In}(s) - \text{pos}(s)) + RCOMP(s) \cdot T_D(s)$$

or solving for  $\text{pos}(s)$

$$\text{pos}(s) = \frac{1}{s} \cdot \frac{RCLTF(s) \cdot C_P(s)}{1 + \frac{1}{s} \cdot RCLTF(s) \cdot C_P(s)} \text{Pos Cmd In}(s) + \frac{1}{s} \cdot \frac{RCOMP(s)}{1 + \frac{1}{s} \cdot RCLTF(s) \cdot C_P(s)} \cdot T_D(s)$$

The position open loop transfer function is given by:



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$$POLTF(s) = \frac{1}{s} \cdot RCLTF(s) \cdot C_p(s)$$

$$= RCLTF(s) \frac{2 \cdot \pi \cdot 20 \cdot 2 \cdot \pi \cdot 160 \cdot (s + 2 \cdot \pi \cdot 10)}{s^2 \cdot (s + 2 \cdot \pi \cdot 160)}$$

It can be seen from Figure 16 that over the position loop bandwidth the closed rate loop magnitude is ~1, however there is a small phase lag contribution from this loop. The Bode plot for the position loop is shown in Figure 17 which indicates the gain crossover frequency is 24 Hz, the PM is 54°, the GM is 13 dB, and the phase crossover is 90 Hz.

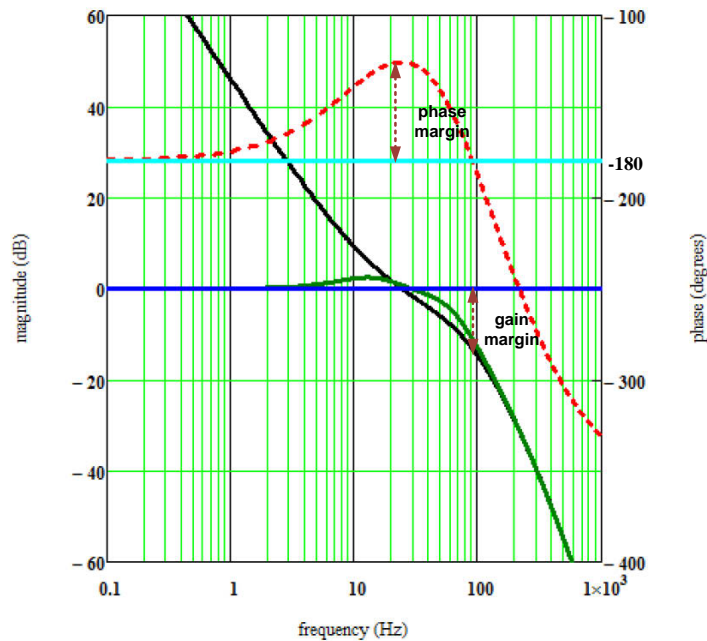


Figure 17 Bode plot for outer position loop

The expression for the torque compliance is obtained as:

$$PCOMP(s) = \frac{1}{s} \cdot \frac{RCOMP(s)}{1 + \frac{1}{s} \cdot RCLTF(s) \cdot C_p(s)} = \frac{1}{s} \cdot \frac{(1 + ROLTF(s)) \cdot RCOMP(s)}{1 + \frac{1}{s} \cdot ROLTF(s) \cdot (s + C_p(s))}$$

Noting that

$$RCOMP(s) = \frac{1}{J \cdot s} \cdot \frac{1}{1 + ROLTF(s)}$$

The expression for the compliance can be expressed as:



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$$PCOMP(s) = \frac{1}{J \cdot s^2} \cdot \frac{1}{1 + \frac{1}{s} \cdot ROLTF(s) \cdot (s + C_p(s))}$$

There are several advantages to the inner/outer loop design. The inner rate loop provides a damping effect for the position loop response. If positioning is implemented with a sensor such as a camera (a simple model being the outer loop summing junction) to track a target, the loop is referred to as the track loop. Using the inner/outer loop architecture allows the track sensor or camera to operate over lower bandwidth reducing noise on the track signal while the higher bandwidth rate loop primarily rejects disturbances. Actually the effective gain and thereby disturbance rejection is a function of the product of the open loop gain for both loops, but the inner rate loop gain normally dominates. The compliance curve for this design is shown in Figure 18. Comparing the rejection at 10 Hz with the compliance for the first example shown in Figure 14, this design provides ~5x (16 dB vs 30 dB) the rejection of the single loop design.

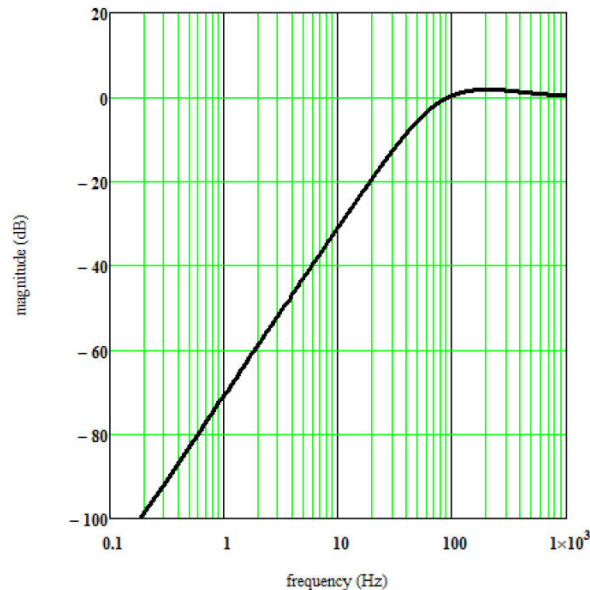


Figure 18 Torque Compliance for Motion Control Example 2

**10 Digital Control:** The majority of control algorithms today are implemented in an embedded or digital signal processor. Signals to be measured are sampled at a defined rate and commands to control the plant or load are transmitted at the same rate. The motion control examples discussed previously will be used to illustrate the digital design procedure. This sample data control configuration is illustrated for the Motion Control Example 1 configuration in Figure 19. The block diagram is modified to include sample functions which are analog to digital converters (ADC) or digital to analog converters (DAC) that sample signals or generate commands at a



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specified  $\Delta T$  time interval. Mathematically these devices can be modeled as zero order hold (ZOH) functions which sample an input or apply an output at a specified time increment; holding that value until the next time increment. Inherent with this approach is internal sampling within the processor.

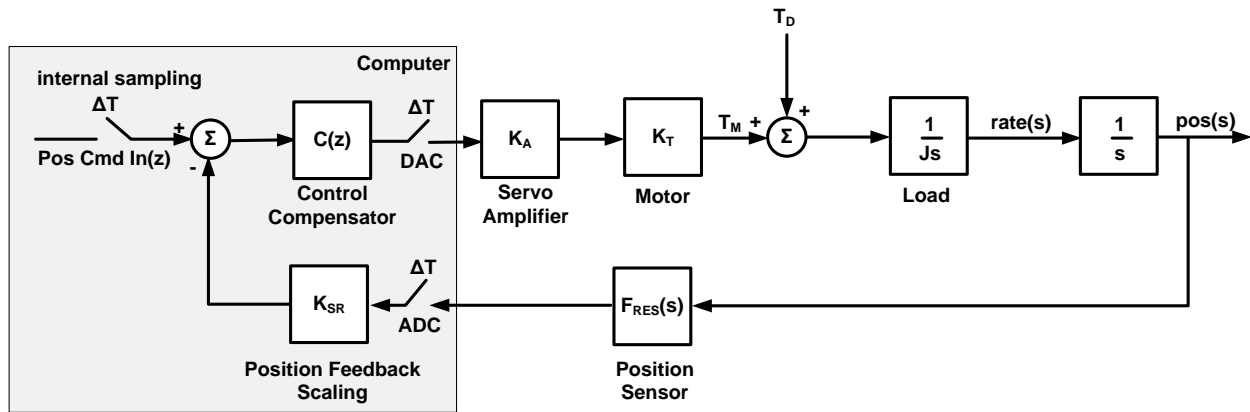


Figure 19 Direct LOS Stabilization Control Configuration with a Digital Controller

With sampling, the infinite continuous frequency spectrum is mapped to a spectrum whose maximum frequency is half the sampling frequency per Nyquist's theorem. So the compensator  $C(s)$  must be converted to a digital format; transforming it from a continuous frequency dependent function into the sampled frequency spectrum limited at half the sampling frequency. Many textbooks are available on the theory of sample-data systems as is an excellent SunCam Course EE 060 "Converting Feedback Systems from Analog to Digital Control". The  $z$ -transform is often used to represent the frequency response of transfer functions in the sample data spectrum. The bilinear transformation with frequency pre-warping is one method for transforming a transfer function in the  $s$ -domain to one in the  $z$ -domain. For a specified continuous transfer function  $G(s)$ , the discrete equivalent can be determined from:

$$G(z) = G(s) \Big|_{s = \frac{2}{\Delta T} \frac{z-1}{z+1}}$$

Frequency pre-warping tries to compensate for the frequency distortion that occurs when mapping the infinite continuous frequency spectrum into a sample limited frequency spectrum using this transformation. Critical frequencies in the continuous domain are modified as:

$$f_{\text{sampled}} = \frac{1}{\pi \cdot \Delta T} \tan(\pi \cdot f_{\text{continuous}} \cdot \Delta T)$$

As an example, the motion control example 1 compensator  $C(s)$  was given by:

$$C(s) = \frac{K_A \cdot K_T}{J} \cdot \frac{2 \cdot \pi \cdot 40 \cdot 2 \cdot \pi \cdot 160 \cdot (s + 2 \cdot \pi \cdot 16)}{(s + 2 \cdot \pi \cdot 160)}$$



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The sampling frequency is assumed to be 4000 Hz ( $\Delta T=0.00025$  sec). There are three critical frequencies, when pre-warped become:

$$2 \cdot \pi \cdot 40 \Rightarrow \omega_{s1} = 2 \cdot \pi \cdot 39.986$$

$$2 \cdot \pi \cdot 16 \Rightarrow \omega_{s2} = 2 \cdot \pi \cdot 15.999$$

$$2 \cdot \pi \cdot 160 \Rightarrow \omega_{s3} = 2 \cdot \pi \cdot 159.165$$

Because of the high sample rate relative to the critical frequencies, none really change significantly from their continuous frequency equivalent. In the z-domain using the bilinear transform the compensator is given by:

$$C(z) = C(s) \Big|_{s=\frac{2}{\Delta T} \frac{z-1}{z+1}} = K_0 \cdot \left[ \frac{z-\alpha}{z-\beta} \right]$$

$$\text{with } K_0 = \frac{K_A \cdot K_T}{J} \cdot \omega_{s1} \cdot \omega_{s3} \cdot \left[ \frac{1 + \frac{\Delta T \cdot \omega_{s2}}{2}}{1 + \frac{\Delta T \cdot \omega_{s3}}{2}} \right]; \quad \alpha = \frac{1 - \frac{\Delta T \cdot \omega_{s2}}{2}}{1 + \frac{\Delta T \cdot \omega_{s2}}{2}}; \quad \beta = \frac{1 - \frac{\Delta T \cdot \omega_{s3}}{2}}{1 + \frac{\Delta T \cdot \omega_{s3}}{2}}$$

For digital implementation, this can be expressed as:

$$C(z^{-1}) = K_0 \frac{1 - \alpha \cdot z^{-1}}{1 - \beta \cdot z^{-1}}$$

Finally in digital format, for a compensator input X at sample interval k, the output Y at k can be expressed as:

$$Y(k) = \alpha \cdot Y(k-1) + K_0 \cdot (X(k) - \alpha \cdot X(k-1))$$

The Bode plot for the sampled system is shown in Figure 20 and can be compared to Figure 13 for the continuous time version. Due to the sampling, the phase margin is reduced to  $51^\circ$  while the gain margin is no longer infinite but  $\sim 28$  dB.





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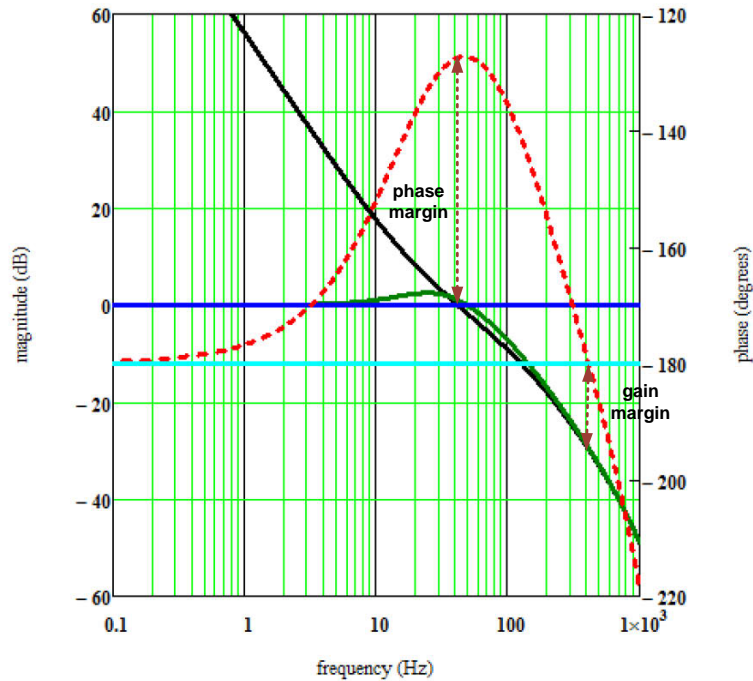


Figure 20 Bode plot for Sampled Version of Motion control Example 1

The motion feedback control loop for example 2 with rate plus position feedback, implemented as a sampled controller, is shown in Figure 21.

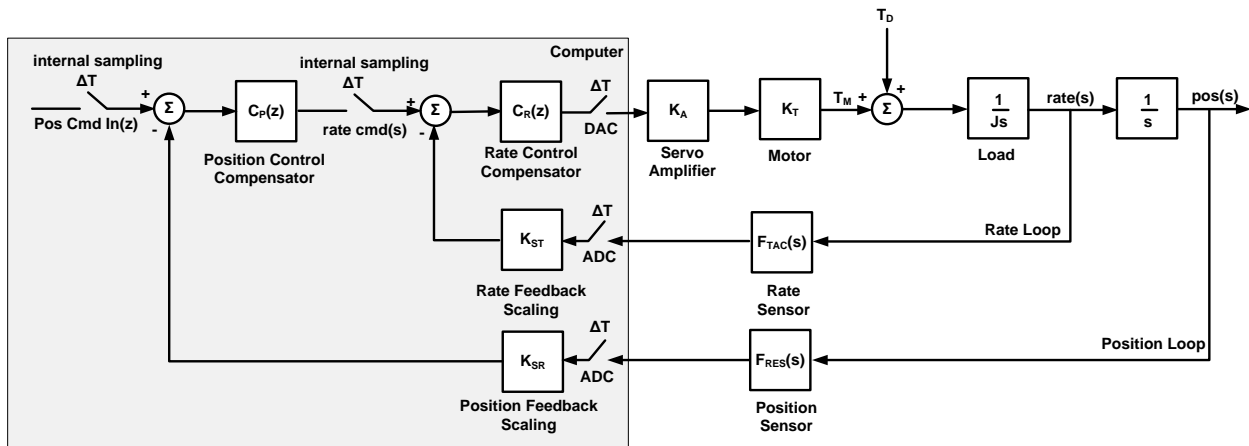


Figure 21 Sample Data Motion Control with Position and Rate Feedback

The rate open loop transfer function and control compensator were given by:



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$$\begin{aligned}
 ROLTF(s) &= \frac{K_A \cdot K_T}{J \cdot s} \cdot C_R(s) \\
 &= \frac{K_A \cdot K_T}{J \cdot s} \cdot \frac{K_S \cdot 2 \cdot \pi \cdot 100 \cdot 2 \cdot \pi \cdot 400 \cdot (s + 2 \cdot \pi \cdot 35)}{s \cdot (s + 2 \cdot \pi \cdot 400)} \\
 &= \frac{2 \cdot \pi \cdot 100 \cdot 2 \cdot \pi \cdot 400 \cdot (s + 2 \cdot \pi \cdot 35)}{s^2 \cdot (s + 2 \cdot \pi \cdot 400)}
 \end{aligned}$$

where

$$C_R(s) = \frac{2 \cdot \pi \cdot 100 \cdot 2 \cdot \pi \cdot 400 \cdot (s + 2 \cdot \pi \cdot 35)}{s \cdot (s + 2 \cdot \pi \cdot 400)}$$

Using the same bilinear transformation with frequency pre-warping as described for the last example and the sampling frequency of 4000 Hz ( $\Delta T = 0.00025$  sec) the three critical frequencies, when pre-warped are:

$$\begin{aligned}
 2 \cdot \pi \cdot 100 &\Rightarrow \omega_{s1} = 2 \cdot \pi \cdot 99.795 \\
 2 \cdot \pi \cdot 35 &\Rightarrow \omega_{s2} = 2 \cdot \pi \cdot 34.99 \\
 2 \cdot \pi \cdot 400 &\Rightarrow \omega_{s3} = 2 \cdot \pi \cdot 387.567
 \end{aligned}$$

Again because of the high sample rate relative to the critical frequencies, only the frequency at 400 Hz changed slightly from their continuous frequency equivalent. In the z-domain using the bilinear transform, the compensator is expressed as:

$$C_R(z) = C_R(s) \Big|_{s=\frac{2}{\Delta T} \frac{z-1}{z+1}} = K_{0R} \cdot \left[ \frac{z - \alpha_R}{z - 1} \right] \cdot \left[ \frac{z + 1}{z - \beta_R} \right]$$

with 
$$K_{0R} = \frac{K_A \cdot K_T}{J} \cdot \omega_{s2} \cdot \left[ 1 + \frac{\Delta T \cdot \omega_{s1}}{2} \right] \cdot \left[ \frac{\frac{\Delta T \cdot \omega_{s3}}{2}}{1 + \frac{\Delta T \cdot \omega_{s3}}{2}} \right] ; \quad \alpha_R = \frac{1 - \frac{\Delta T \cdot \omega_{s1}}{2}}{1 + \frac{\Delta T \cdot \omega_{s1}}{2}} ; \quad \beta_R = \frac{1 - \frac{\Delta T \cdot \omega_{s3}}{2}}{1 + \frac{\Delta T \cdot \omega_{s3}}{2}}$$

The Bode plot for the rate feedback loop is shown in Figure 22. The phase margin is  $\sim 48^\circ$  and the gain margin 22 dB. The plot of the higher bandwidth loop also shows the effect of the 4000 Hz sample function.



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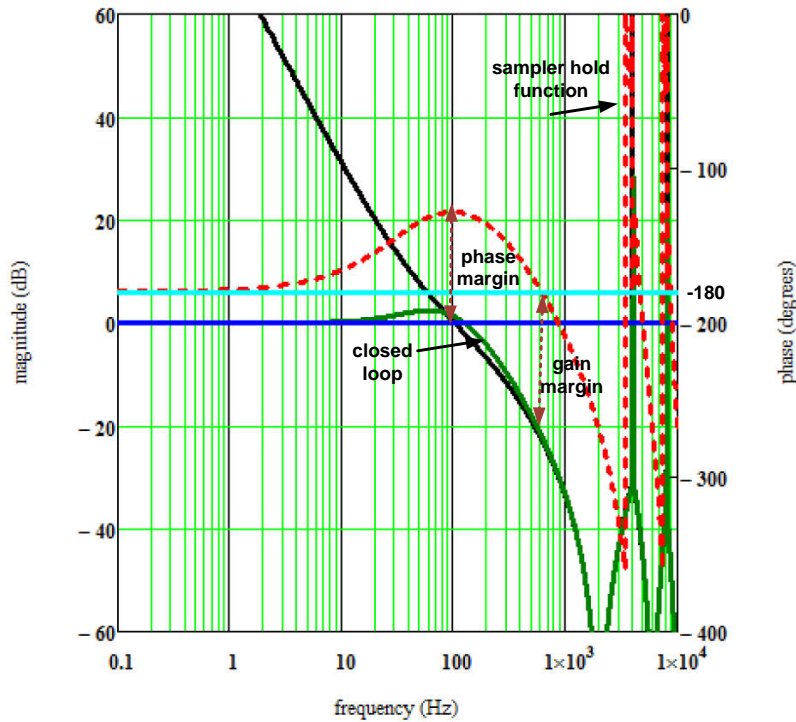


Figure 22 Bode plot for Sample Data Rate Feedback Loop

The compensator for the position feedback loop was given by:

$$C_P(s) = \frac{2 \cdot \pi \cdot 20 \cdot 2 \cdot \pi \cdot 160 \cdot (s + 2 \cdot \pi \cdot 10)}{s \cdot (s + 2 \cdot \pi \cdot 160)}$$

The same process is followed again with:

$$2 \cdot \pi \cdot 20 \Rightarrow \omega_{s1} = 2 \cdot \pi \cdot 19.998$$

$$2 \cdot \pi \cdot 10 \Rightarrow \omega_{s2} = 2 \cdot \pi \cdot 9.999$$

$$2 \cdot \pi \cdot 160 \Rightarrow \omega_{s3} = 2 \cdot \pi \cdot 159.165$$

The position compensator in the z-domain can then be expressed as:

$$C_P(z) = C_P(s) \Big|_{s=\frac{2}{\Delta T} \frac{z-1}{z+1}} = K_{0P} \cdot \left[ \frac{z - \alpha_P}{z - 1} \right] \cdot \left[ \frac{z + 1}{z - \beta_P} \right]$$

$$\text{with } K_{0P} = \omega_{s2} \cdot \left[ 1 + \frac{\Delta T \cdot \omega_{s1}}{2} \right] \cdot \left[ \frac{\frac{\Delta T \cdot \omega_{s3}}{2}}{1 + \frac{\Delta T \cdot \omega_{s3}}{2}} \right] ; \alpha_P = \frac{1 - \frac{\Delta T \cdot \omega_{s1}}{2}}{1 + \frac{\Delta T \cdot \omega_{s1}}{2}} ; \beta_P = \frac{1 - \frac{\Delta T \cdot \omega_{s3}}{2}}{1 + \frac{\Delta T \cdot \omega_{s3}}{2}}$$



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The Bode plot for the position plus rate feedback control is shown in Figure 23. The phase margin is  $36^\circ$  and gain margin 12.5 dB. There is also significant closed loop peaking. Additional compensation should be applied to the loop to improve the response.

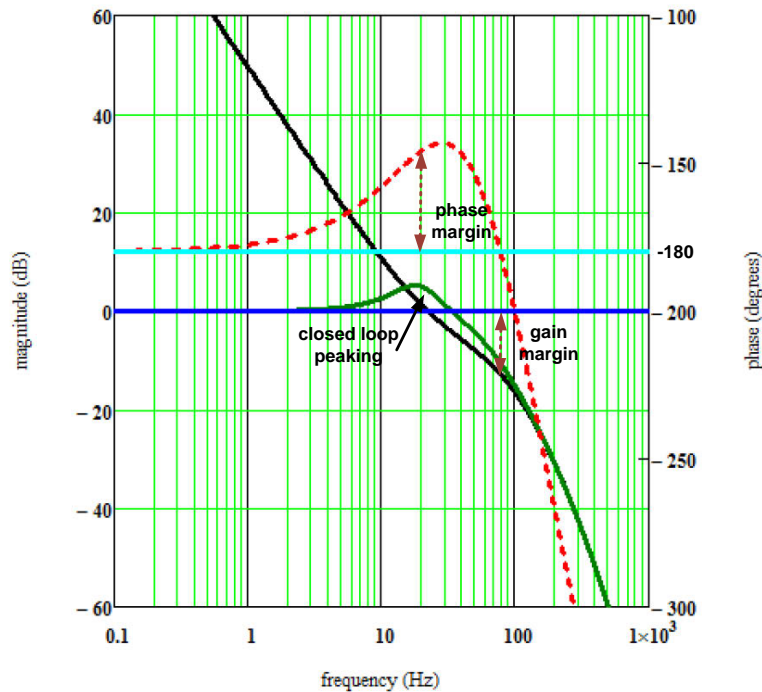


Figure 23 Bode plot for Sample Data Motion Control System; Position and Rate Feedback Loops

**11 Summary:** This completes course material. The topics covered began with a description of the basic block diagram in section 2. The relationships between time and frequency domain representations of the block diagram elements were discussed in section 3 followed by the key feedback relationships derived from the block diagram algebra in section 4. Control loop stability and methods to determine stability margins were described in section 5 followed by a discussion of specifying control loop performance in section 6. A couple of control loop design methods were provided in section 7. The basic theory was then applied to two examples; a home heating system in section 8 and motion control applications in section 9. Converting to a digital sample data controller was discussed in section 10; as related to the motion control example in section 9.