Digital Control of Second and Higher Order Systems©

By

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1.0 Introduction -

The availability of inexpensive microprocessor-based embedded digital controllers has allowed the use of digital control techniques in modern systems. Digital controllers do not exhibit the drift and temperature dependence of their analog counterparts.

"Modern Controls Theory" permits a direct design of a digital controller without first providing an analog controller counterpart. One such approach is the "deadbeat" controller, for which there is no analog equivalent. We use a deadbeat example to show that a relatively fast controller can be constructed for high-order systems without first designing an analog controller, nor resorting to complicated pole-placement in the unit-circle "z" domain.

2.0 A Third-Order System Example –

We have chosen a third-order system $\boldsymbol{H(f) = H_1(f) * H_2(f)}$ with one real pole $\boldsymbol{H_1(f)}$ and a complex-conjugate pole-pair $\boldsymbol{H_2(f)}$. We choose the complex pole pair so that we face the control of an under damped system.

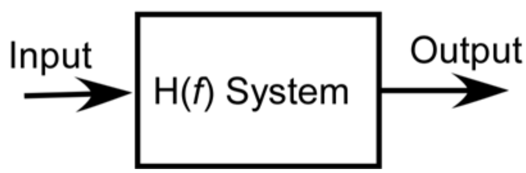

Figure 2.0 –Example Third-Order System to be Controlled

The first-order pole defining equation follows as:

$$H_1(f) = \frac{1}{\left[1 + j\left(\frac{f}{f_1}\right)\right]} \qquad [2.0]$$

The second-order pole defining equation follows as:

$$H_2(f) = \frac{1}{\left[1 - \left(\frac{f}{f_2}\right)^2\right] + j\left[2\zeta\left(\frac{f}{f_2}\right)\right]} \qquad [2.1]$$

The real pole is chosen to be placed f_1 = 1.6 Hertz and the complex pole-pair at f_2 = 16 Hertz, a decade higher in frequency. We have chosen the damping factor ζ = 0.25 so that we emphasize the under damped component.

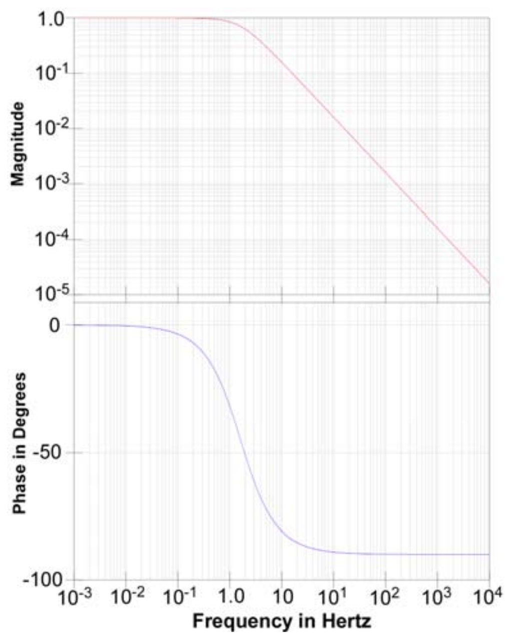

Figure 2.1 – Bode Plot of Single Pole $H_1(f)$ Response

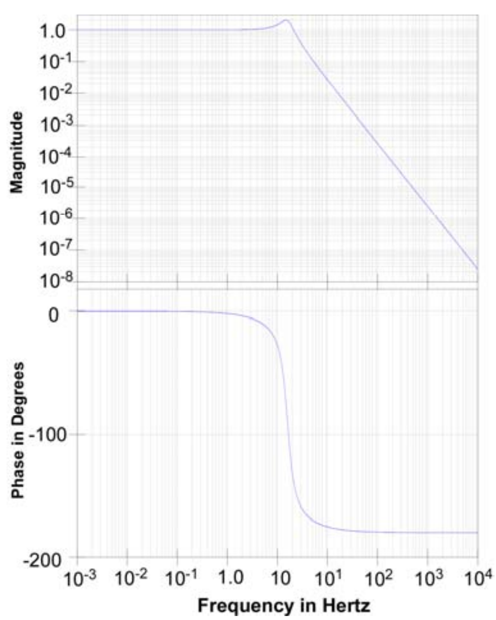

Figure 2.2 – Bode Plot of Complex Pole-Pair $H_2(f)$ Response

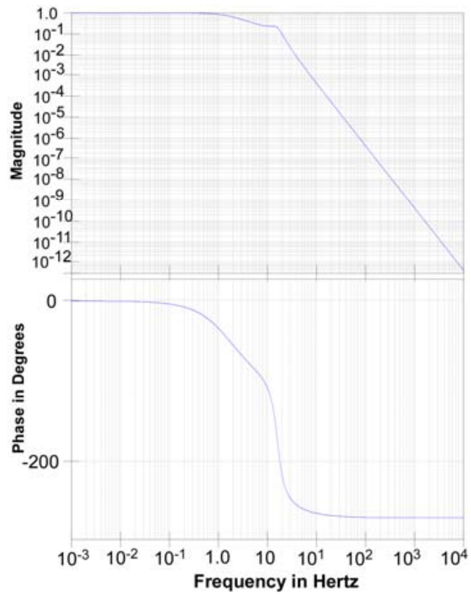

Figure 2.3 – Bode Plot of Third-Order *H(f) = H₁(f)* H₂(f)* System Response

3.0 A Third-Order System Block Diagram–

We represent the third-order system in the following canonical form so that we can extract a polynomial representation of the transfer function ***H(s)*** that is equivalent to the ***H(f)*** we have constructed.

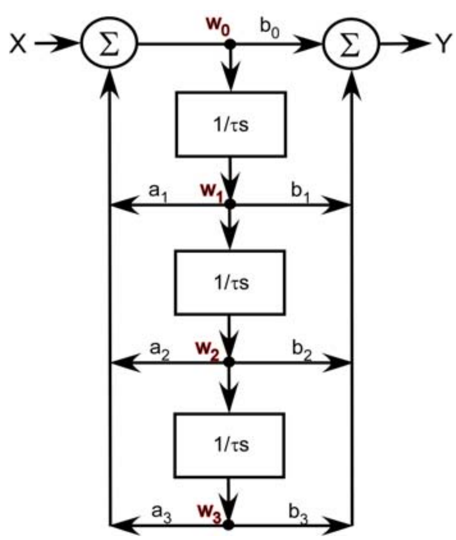

Figure 3.0 – Canonical Third-Order System Block Diagram

We express our transfer functions in a rational polynomial form by restating each component in Laplace transform notation, using the scaled τ***s*** operator.

The first-order pole ***H₁(s)*** defining equation follows as:

$$H_1(s)=\frac{1}{\left[1+\left(\frac{1}{2\pi f_1}\right)s\right]} \qquad [3.0]$$

The second-order pole ***H₂(s)*** defining equation follows as:

$$H_2(s)=\frac{1}{\left[1+2\zeta\left(\frac{1}{2\pi f_2}\right)s+\left(\frac{1}{2\pi f_2}\right)^2 s^2\right]} \qquad [3.1]$$

The third-order defining equation ***H(s)*** follows as:

$$H(s)=H_1(s)H_2(s)=\frac{1}{1+\frac{s}{2\pi f_1}}\bullet\frac{1}{1+2\zeta\frac{s}{2\pi f_2}+\left(\frac{s}{2\pi f_2}\right)^2} \qquad [3.2]$$

$$H(s)=\frac{1}{\left[1+2\zeta\frac{s}{2\pi f_2}+\left(\frac{s}{2\pi f_2}\right)^2\right]+\frac{s}{2\pi f_1}\left[1+2\zeta\frac{s}{2\pi f_2}+\left(\frac{s}{2\pi f_2}\right)^2\right]} \qquad [3.3]$$

$$H(s)=\frac{1}{\left[1+\left(\frac{1}{2\pi}\right)\left(\frac{2\zeta}{f_2}+\frac{1}{f_1}\right)s+\left(\frac{1}{2\pi}\right)^2\left(\frac{1}{f_2^2}+\frac{2\zeta}{f_1 f_2}\right)s^2+\left(\frac{1}{2\pi}\right)^3\left(\frac{1}{f_1 f_2^2}\right)s^3\right]} \qquad [3.4]$$

Equation 3.4 is the defining equation that we must map onto the equivalent canonical third-order system block diagram.

We write the summation equations, the integrator relationships, and we solve for the transfer function. Finally, we map the coefficients of powers of the ***s*** operator and the Block diagram is parameterized, as follows:

$$w_0 = a_1 w_1 + a_2 w_2 + a_3 w_3 + X \qquad [3.5]$$

$$Y = b_0 w_0 + b_1 w_1 + b_2 w_2 + b_3 w_3 \qquad [3.6]$$

$$w_3 = \frac{w_2}{\tau s} \Rightarrow w_2 = \tau w_3 s \qquad [3.7]$$

$$w_2 = \frac{w_1}{\tau s} \Rightarrow w_1 = \tau w_2 s = w_3 \tau^2 s^2 \qquad [3.8]$$

$$w_1 = \frac{w_0}{\tau s} \Rightarrow w_0 = \tau w_1 s = w_3 \tau^3 s^3 \qquad [3.9]$$

We substitute equations [3.7], [3.8], and [3.9] into equation [3.5] as follows:

$$w_0 = w_3 \tau^3 s^3 = a_1 w_3 \tau^2 s^2 + a_2 w_3 \tau s + a_3 w_3 + X \qquad [3.10]$$

$$w_3 \tau^3 s^3 - a_1 w_3 \tau^2 s^2 - a_2 w_3 \tau s - a_3 w_3 = X \qquad [3.11]$$

$$\left[\tau^3 s^3 - a_1 \tau^2 s^2 - a_2 \tau s - a_3\right] w_3 = X \qquad [3.12]$$

$$w_3 = \frac{X}{\left[\tau^3 s^3 - a_1 \tau^2 s^2 - a_2 \tau s - a_3\right]} \qquad [3.13]$$

From equation 3.13, we find:

$$w_2 = w_3 \tau s = \frac{X \tau s}{\left[\tau^3 s^3 - a_1 \tau^2 s^2 - a_2 \tau s - a_3\right]} \qquad [3.14]$$

$$w_1 = w_3 \tau^2 s^2 = \frac{X \tau^2 s^2}{\left[\tau^3 s^3 - a_1 \tau^2 s^2 - a_2 \tau s - a_3\right]} \qquad [3.15]$$

$$w_0 = w_3 \tau^3 s^3 = \frac{X \tau^3 s^3}{\left[\tau^3 s^3 - a_1 \tau^2 s^2 - a_2 \tau s - a_3\right]} \qquad [3.16]$$

Finally, we use equation [3.6] to generate:

$$Y = \frac{\left[b_0\tau^3s^3 + b_1\tau^2s^2 + b_2\tau s + b_3\right]}{\left[\tau^3s^3 - a_1\tau^2s^2 - a_2\tau s - a_3\right]}X \qquad [3.17]$$

From equation 3.8, we see that our third-order system has the parameters:

$$b_0 = b_1 = b_2 = 0 \qquad [3.18]$$

$$Y = \frac{\left[b_3\right]}{\left[\tau^3s^3 - a_1\tau^2s^2 - a_2\tau s - a_3\right]}X = \frac{1}{\left[\frac{1}{b_3}\tau^3s^3 - \frac{a_1}{b_3}\tau^2s^2 - \frac{a_2}{b_3}\tau s - \frac{a_3}{b_3}\right]}X \qquad [3.19]$$

$$\frac{Y}{X} = H(s) = \frac{1}{\left[\frac{1}{b_3}\tau^3s^3 - \frac{a_1}{b_3}\tau^2s^2 - \frac{a_2}{b_3}\tau s - \frac{a_3}{b_3}\right]} \qquad [3.20]$$

$$\frac{Y}{X} = H(s) = \frac{1}{\left[\frac{1}{8\pi^3 f_1 f_2^2}s^3 + \frac{1}{4\pi^2}\left(\frac{1}{f_2^2} + \frac{2\zeta}{f_1 f_2}\right)s^2 + \frac{1}{2\pi}\left(\frac{2\zeta}{f_2} + \frac{1}{f_1}\right)s + 1\right]} \qquad [3.21]$$

We set coefficients of the powers of ***s*** equal and solve, as follows:

$$\frac{\tau^3}{b_3} = \frac{1}{8\pi^3 f_1 f_2^2} \Rightarrow b_3 = 8\pi^3\tau^3 f_1 f_2^2 \qquad [3.22]$$

$$-\frac{a_1}{b_3}\tau^2 = \frac{1}{4\pi^2}\left(\frac{1}{f_2^2} + \frac{2\zeta}{f_1 f_2}\right)$$

$$a_1 = -\frac{8\pi^3\tau^3 f_1 f_2^2}{4\pi^2\tau^2}\left(\frac{1}{f_2^2} + \frac{2\zeta}{f_1 f_2}\right) = -2\pi\tau\left(f_1 + 2\zeta f_2\right) \qquad [3.23]$$

$$-\frac{a_2}{b_3}\tau = \frac{1}{2\pi}\left(\frac{2\zeta}{f_2} + \frac{1}{f_1}\right)$$

$$a_2 = -\frac{8\pi^3\tau^3 f_1 f_2^2}{2\pi\tau}\left(\frac{2\zeta}{f_2}+\frac{1}{f_1}\right) = -4\pi^2\tau^2\left(2\zeta f_1 f_2 + f_2^2\right) \qquad [3.24]$$

$$-\frac{a_3}{b_3} = 1 \Rightarrow a_3 = -8\pi^3\tau^3 f_1 f_2^2 \qquad [3.25]$$

F_1	F_2	ζ	π	τ
1.6	16	0.25	3.141593	0.01

Coefficients

a_1	**-0.603186**
a_2	**-1.061180**
a_3	**-0.101601**
b_3	**0.101601**

Table 3.0 –Third-Order System Block Diagram Parameters

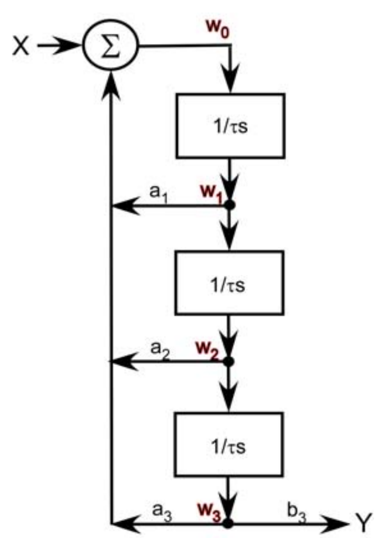

Figure 3.1 – Parameterized Third-Order System Block Diagram

4.0 Step and Impulse Responses of the Models–

We simulate the analog system step responses of the components and compare results in the following:

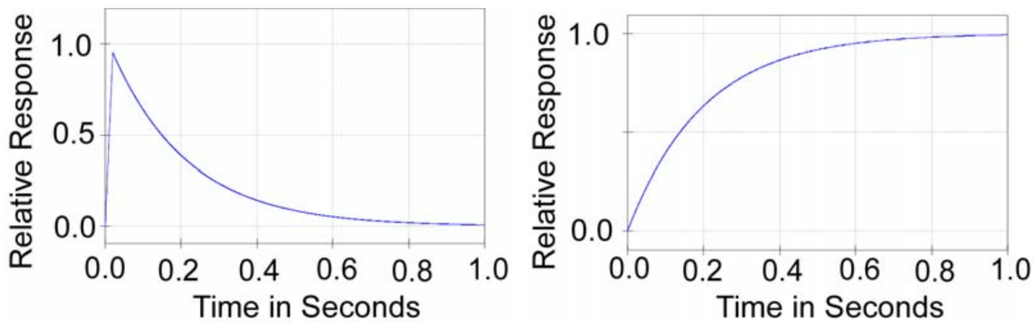

Figure 4.0 – First-Order System Impulse and Step Response

Figure 4.0 illustrates the time-domain responses of equation [3.0] for the first-order component realization alone.

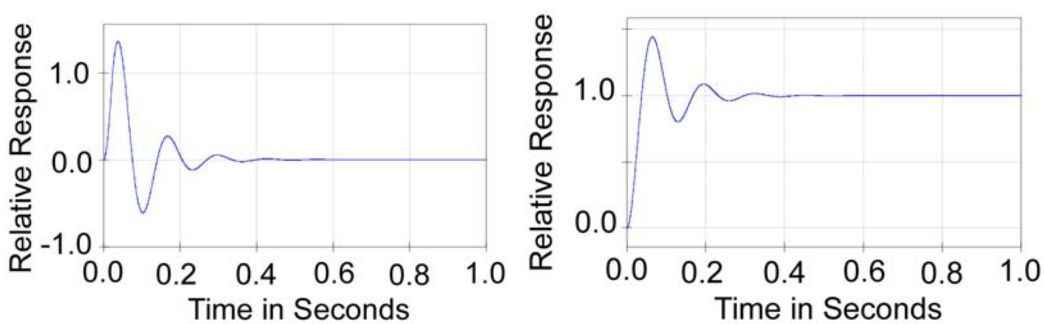

Figure 4.1 – Second-Order System Impulse and Step Response

Figure 4.1 illustrates the time-domain responses of equation [3.1] for the second-order component realization alone.

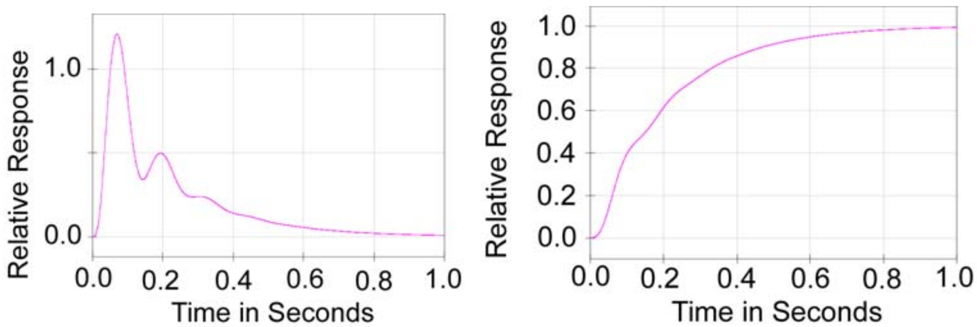

Figure 4.2 – Third-Order System Impulse and Step Response

Figure 4.2 illustrates the time-domain responses of cascades with ***$H_1(s)$*** followed by ***$H_2(s)$***, and with ***$H_2(s)$*** followed by ***$H_1(s)$***, as well as the parameterized third-order block alone. At the scale of the presentation tool, the displays of all three are indistinguishable.

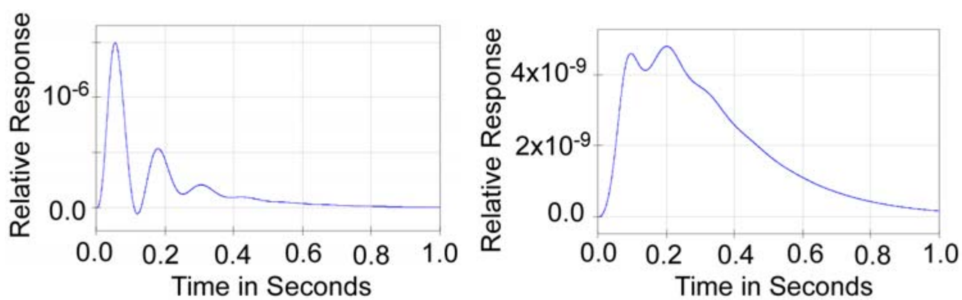

Figure 4.3 – Third-Order Cascade Impulse and Step Response Differences

Figure 4.3 illustrates the time-domain difference of the responses of cascades with ***$H_1(s)$*** followed by ***$H_2(s)$***, and with ***$H_2(s)$*** followed by ***$H_1(s)$***. The blocks were copied in each order from the single instance models and the differences noted are numerical errors of the simulation engine.

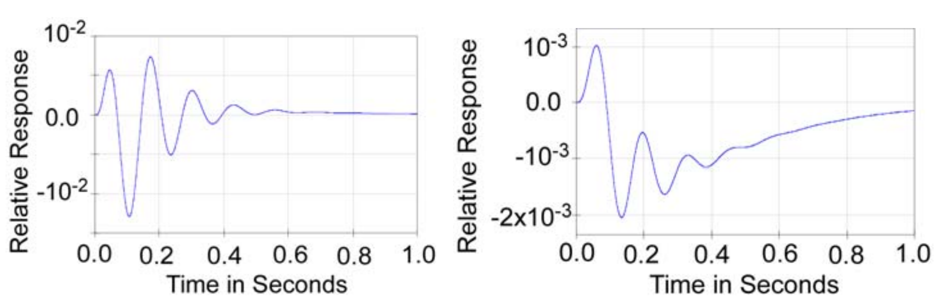

Figure 4.4 – Third-Order Impulse and Step Response Differences

Figure 4.4 illustrates the time-domain difference of the responses of cascades with respect to the parameterized third-order block model. The differences noted are numerical errors of the parameter values mapped to available component values in the analog simulation engine. The maximum differential error of ~1% is difficult to distinguish with the Impulse responses superimposed, and the maximum differential error of ~0.1% is cannot be distinguished in the superimposed Step responses.

With the parameterization above in Table 3.0, we have an accurate third-order analog model for comparison to later work.

5.0 A Discrete-Time Model Using Shift-Operator Notation–

For a system with a Zero-Order Hold (ZOH) at the input and output, such as encountered with the Analog to Digital and Digital to Analog converters at the interface between the analog model and the digital controller, we can employ the methodology from "Computer Controlled Systems," as taught by the authors, Karl J. Astrom and Bjorn Wittenmark, and by Graham C. Goodwin and Kwai Sang Sin in "Adaptive Filtering, Prediction, and Control."

The methodology is based on the time-shift operator ***q***, and on a "backward-difference" to represent differentiation.

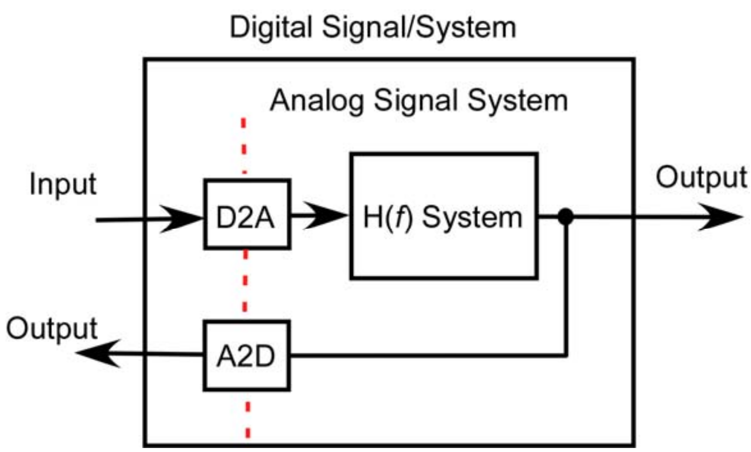

Figure 5.0 – Analog System *H(f)* enclosed in ZOH functions to form Digital *H(q)*

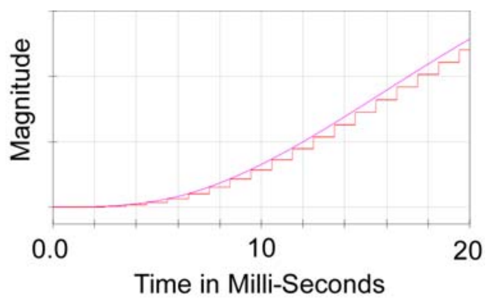

Figure 5.1 – ZOH Output Waveform Representation

From the references, we represent the third-order system in multiple polynomial forms. We present a cascade of first-order system followed by a second-order system for a composite third-order system, a cascade of second-order system followed by a first-order system for a composite third-order system, and a polynomial third-order system. We show that they produce the same results, but are distinct in terms of the algebra of the input/output transfer function relationship.

For the single pole, a relatively simple substitution follows:

$$H_1(s)=\frac{1}{\left[1+\left(\frac{1}{2\pi f_1}\right)s\right]}\Rightarrow H_1(q)=\frac{1-e^{-2\pi f_1 T_s}}{q-e^{-2\pi f_1 T_s}}=\frac{1-a_{11}}{q-a_{11}} \qquad [5.0]$$

$$a_{11}=e^{-2\pi f_1 T_s} \qquad [5.1]$$

We have defined the constant $\boldsymbol{a_{11}}$ to distinguish it from the $\boldsymbol{a_1}$ used later in the second-order model that follows.

For the second-order pole $\boldsymbol{H_2(s)}$ with the defining equation as follows:

$$H_2(s)=\frac{1}{\left[1+2\zeta\left(\frac{1}{2\pi f_2}\right)s+\left(\frac{1}{2\pi f_2}\right)^2 s^2\right]} \qquad [5.2]$$

We refer to the shift-operator methodology's second-order defining equations (presented here without proof) in the companion forms:

$$H_2(s)=\frac{1}{\left[1+2\zeta\left(\frac{1}{\varpi_0}\right)s+\left(\frac{1}{\varpi_0}\right)^2 s^2\right]}\Rightarrow H_2(q)=\frac{b_1 q+b_2}{q^2+a_1 q+a_2} \qquad [5.3]$$

$$\varpi_0 = 2\pi f_2 \qquad [5.4]$$

$$\alpha = e^{-\zeta\varpi_0 T_S} \qquad [5.5]$$

$$\varpi = \varpi_0\sqrt{1-\zeta^2} \qquad [5.6]$$

$$\beta = \cos(\varpi T_S) \qquad [5.7]$$

$$\gamma = \sin(\varpi T_S) \qquad [5.8]$$

$$a_2 = \alpha^2 = e^{-2\zeta\varpi_0 T_S} \qquad [5.9]$$

$$a_1 = -2\alpha\beta = -2e^{-\zeta\varpi_0 T_S}\cos\left(T_S\varpi_0\sqrt{1-\zeta^2}\right) \qquad [5.10]$$

$$b_1 = 1-\alpha\left(\frac{\zeta\varpi_0}{\varpi}\gamma+\beta\right)\Rightarrow$$

$$b_1 = 1-e^{-\zeta\varpi_0 T_S}\left(\frac{\zeta\varpi_0}{\varpi_0\sqrt{1-\zeta^2}}\sin\left(\varpi_0 T_S\sqrt{1-\zeta^2}\right)+\cos\left(\varpi_0 T_S\sqrt{1-\zeta^2}\right)\right) \qquad [5.11]$$

$$b_2 = \alpha^2+\alpha\left(\frac{\zeta\varpi_0}{\varpi}\gamma-\beta\right)\Rightarrow$$

$$b_2 = e^{-2\zeta\varpi_0 T_S}+e^{-\zeta\varpi_0 T_S}\left(\frac{\zeta\varpi_0}{\varpi_0\sqrt{1-\zeta^2}}\sin\left(\varpi_0 T_S\sqrt{1-\zeta^2}\right)-\cos\left(\varpi_0 T_S\sqrt{1-\zeta^2}\right)\right) \qquad [5.12]$$

From our known model ***H(q) = H₁(q)* H₂(q)*** and we have the function as follows:

$$H_1(q)=\frac{Y_1(q)}{X_1(q)}=\frac{1-a_{11}}{q-a_{11}} \qquad [5.13]$$

$$[q - a_{11}]Y_1(q) = (1 - a_{11})X_1(q) \qquad [5.14]$$

$$qY_1(q) = a_{11}Y_1(q) + (1 - a_{11})X_1(q) \qquad [5.15]$$

Equation [5.15] is true for all instants and samples, so we can multiply both sides by $\boldsymbol{q^{-1}}$ to obtain:

$$Y_1(q) = a_{11}q^{-1}Y_1(q) + (1 - a_{11})q^{-1}X_1(q) \qquad [5.16]$$

Equation [5.16] indicates that each sample of the response of $\boldsymbol{H_1(q)}$ is determined by a combination of one weighted previous sample of $\boldsymbol{Y_1(q)}$ and $\boldsymbol{X_1(q)}$.

In a similar fashion, we develop the model for $\boldsymbol{H_2(q)}$ as follows:

$$H_2(q) = \frac{Y_2(q)}{X_2(q)} = \frac{b_1q + b_2}{q^2 + a_1q + a_2} \qquad [5.17]$$

$$[q^2 + a_1q + a_2]Y_2(q) = [b_1q + b_2]X_2(q) \qquad [5.18]$$

$$q^2Y_2(q) = -[a_1q + a_2]Y_2(q) + [b_1q + b_2]X_2(q) \qquad [5.19]$$

Equation [5.19] is true for all instants and samples, so we can multiply both sides by $\boldsymbol{q^{-2}}$ to obtain:

$$Y_2(q) = -[a_1q + a_2]q^{-2}Y_2(q) + [b_1q + b_2]q^{-2}X_2(q) \qquad [5.20]$$

$$Y_2(q) = -[a_1q^{-1} + a_2q^{-2}]Y_2(q) + [b_1q^{-1} + b_2q^{-2}]X_2(q) \qquad [5.21]$$

Equation [5.21] indicates that each sample of the response of $\boldsymbol{H_2(q)}$ is determined by a combination of two weighted previous samples of $\boldsymbol{Y_2(q)}$ and $\boldsymbol{X_2(q)}$.

6.0 Implementation Model Using Shift-Operator Notation–

We choose to illustrate the Shift-Operator System as two cascaded models first. We will show that placement is unimportant and that $\boldsymbol{H(q) = H_1(q) * H_2(q) = H_2(q) * H_1(q)}$, but we cannot use either form for our "deadbeat" controller..

By placement of $\boldsymbol{H_1(q)}$ first, its output samples are the input samples for $\boldsymbol{H_2(q)}$, and therefore:

$$Y_1(q) = X_2(q) \qquad [6.0]$$

We make the substitution as follows:

$$Y_2(q) = -\left[a_1 q^{-1} + a_2 q^{-2}\right]Y_2(q) + \left[b_1 q^{-1} + b_2 q^{-2}\right]\left[a_{11} q^{-1} Y_1(q) + (1 - a_{11}) q^{-1} X_1(q)\right] \qquad [6.1]$$

$$\begin{aligned} Y(q) = -\left[a_1 q^{-1} + a_2 q^{-2}\right]Y(q) + a_{11}\left[b_1 q^{-2} + b_2 q^{-3}\right]Y_1(q) \\ + (1 - a_{11})\left[b_1 q^{-2} + b_2 q^{-3}\right]X(q) \end{aligned} \qquad [6.2]$$

We note that we still have an explicit reference in equation [6.2] involving the samples of the first-order system $\boldsymbol{Y_1(q)}$, and no way to express that dependence in terms of the third-order sequence $\boldsymbol{Y(q)}$.

By placement of $\boldsymbol{H_2(q)}$ first instead, its output samples are the input samples for $\boldsymbol{H_1(q)}$, and therefore:

$$Y_2(q) = X_1(q) \qquad [6.3]$$

We make the substitution as follows:

$$Y_1(q) = a_{11} q^{-1} Y_1(q) + (1 - a_{11}) q^{-1} X_1(q) \qquad [6.4]$$

$$\begin{aligned} Y_1(q) = a_{11} q^{-1} Y_1(q) \\ -(1 - a_{11}) q^{-1}\left[a_1 q^{-1} + a_2 q^{-2}\right]Y_2(q) + (1 - a_{11}) q^{-1}\left[b_1 q^{-1} + b_2 q^{-2}\right]X_2(q) \end{aligned} \qquad [6.5]$$

$$Y(q) = a_{11} q^{-1} Y(q) - (1 - a_{11})\left[a_1 q^{-2} + a_2 q^{-3}\right]Y_2(q) + (1 - a_{11})\left[b_1 q^{-2} + b_2 q^{-3}\right]X(q) \qquad [6.6]$$

We note that now we have an explicit reference in equation [6.6] involving the samples of the second-order system $\boldsymbol{Y_2(q)}$, and no way to express that in dependence terms of the third-order sequence $\boldsymbol{Y(q)}$.

The explicit reference to the sequence from a cascade member renders a Single-Input/Single-Output (SISO) unavailable. To make a SISO model, we merge the first and second-order cascade to make a single third-order system description before we contrast the three representations.

$$H(q) = \frac{Y(q)}{X(q)} = \frac{(1 - a_{11})\left[b_1 q + b_2\right]}{\left[q - a_{11}\right]\left[q^2 + a_1 q + a_2\right]} \qquad [6.7]$$

$$\frac{Y(q)}{X(q)} = \frac{(1 - a_{11})\left[b_1 q + b_2\right]}{q\left[q^2 + a_1 q + a_2\right] - a_{11}\left[q^2 + a_1 q + a_2\right]} \qquad [6.8]$$

$$\frac{Y(q)}{X(q)}=\frac{(1-a_{11})[b_1q+b_2]}{[q^3+a_1q^2+a_2q]-a_{11}[q^2+a_1q+a_2]} \qquad [6.9]$$

$$\frac{Y(q)}{X(q)}=\frac{(1-a_{11})[b_1q+b_2]}{[q^3+(a_1-a_{11})q^2+(a_2-a_{11}a_1)q-a_{11}a_2]} \qquad [6.10]$$

$$[q^3+(a_1-a_{11})q^2+(a_2-a_{11}a_1)q-a_{11}a_2]Y(q)=(1-a_{11})[b_1q+b_2]X(q) \qquad [6.11]$$

$$q^3Y(q)=-[(a_1-a_{11})q^2+(a_2-a_{11}a_1)q-a_{11}a_2]Y(q)+(1-a_{11})[b_1q+b_2]X(q) \qquad [6.12]$$

We apply the shift operator ***q*** three times, in the form of q^{-3} to equation [6.12] to obtain the present value of ***Y(q)*** in terms of past values of ***Y(q)*** and ***X(q)*** as follows:

$$\begin{aligned}Y(q)=&-[(a_1-a_{11})q^{-1}+(a_2-a_{11}a_1)q^{-2}-a_{11}a_2q^{-3}]Y(q)\\&+(1-a_{11})[b_1q^{-2}+b_2q^{-3}]X(q)\end{aligned} \qquad [6.13]$$

$$\begin{aligned}Y(q)=&[(a_{11}-a_1)q^{-1}+(a_{11}a_1-a_2)q^{-2}+a_{11}a_2q^{-3}]Y(q)\\&+(1-a_{11})[b_1q^{-2}+b_2q^{-3}]X(q)\end{aligned} \qquad [6.14]$$

Equation [6.14] shows us that we can find the response of the modeled system from the past three values of the response and two of the past three samples of the input, despite the rather complicated expression of the model's coefficients. Equations [6.2] and [6.6] also show much the same and we will use simulation to illustrate that they provide the same response. However, equation [6.2] requires the $Y_1(q)$ sequence as an explicit variable, while equation [6.6] requires the $Y_2(q)$ sequence as an explicit variable. In equation [6.14] there are only SISO input and output sampled signals.

We have selected a third-order example with a first order-pole and a complex second order pole because all systems can be decomposed into combinations of these two sets of components. Higher-order systems must be merged into a single-input/single-output (SISO) model such as in equation [6.14], though, for this deadbeat methodology to perform correctly.

In Table 6.0 below, we evaluate each coefficient, and generate the step responses for the single-pole system $Y_1(n)$, the two-pole system $Y_2(n)$, the combined three-pole system ***Y(n)*** as well as the cascade of first and second-order poles in both directions $Y_1(n)$=>$Y_2(n)$ and $Y_2(n)$=>$Y_1(n)$.

We illustrate the responses, as well as the evaluation of difference between the cascade implementations, and also the difference between the third-order system and the cascades. The data was calculated for the first 1000 samples (only the first 10 are shown here). There is appreciable no difference to 10 digits of resolution between the two cascade models, but 1 unit in the 10^{th} digit between the cascade and the third-order model that can be attributed to numerical error.

The spreadsheet was used to check the accuracy of the coefficient calculations because it supports the high precision of calculation to make the accuracy comparison.

F_1	F_2	ζ	π	τ	T_S
1.6	16	0.2500	3.1416	0.0100	0.0001

Shift Operator Model Coefficients

ω_0	100.5310
α	0.989997
ω	97.3387
β	0.999953
γ	0.009734

First-Order Model Coefficient

a_{11}	0.998995

Second-Order Model Coefficient

a_1	-1.979901
a_2	0.980095
b_1	0.007562
b_2	-0.007368

X(n)	n	Y(n)	Y_1(n)	Y_2(n)	Y_a(n) Y_1(n)=>Y_2(n)	Y_b(n) Y_2(n)=>Y_1(n)	Y_a(n)-Y_b(n) Difference	Y(n)-Y_a(n) Difference
0	0	0.000000	0.000000	0.000000				
0	1	0.000000	0.000000	0.000000				
1	2	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1	3	0.000000	0.001005	0.007562	0.000000	0.000000	0.000000	0.000000
1	4	0.000008	0.002009	0.015165	0.000008	0.000008	0.000000	0.000000
1	5	0.000023	0.003011	0.022808	0.000023	0.000023	0.000000	0.000000
1	6	0.000046	0.004013	0.030488	0.000046	0.000046	0.000000	0.000000
1	7	0.000076	0.005014	0.038204	0.000076	0.000076	0.000000	0.000000
1	8	0.000115	0.006014	0.045952	0.000115	0.000115	0.000000	0.000000
1	9	0.000161	0.007012	0.053731	0.000161	0.000161	0.000000	0.000000
1	10	0.000215	0.008010	0.061538	0.000215	0.000215	0.000000	0.000000

Table 6.0 – Model Summary

The spreadsheet comparison in Table 6.0 supports the comparison of digital models. In the illustrations below, we use another, less precise simulation tool to compare the analog third-order model to the digital model. The relative accuracy of the digital model will be used to support its use for making predictions that cannot be made in an analog system.

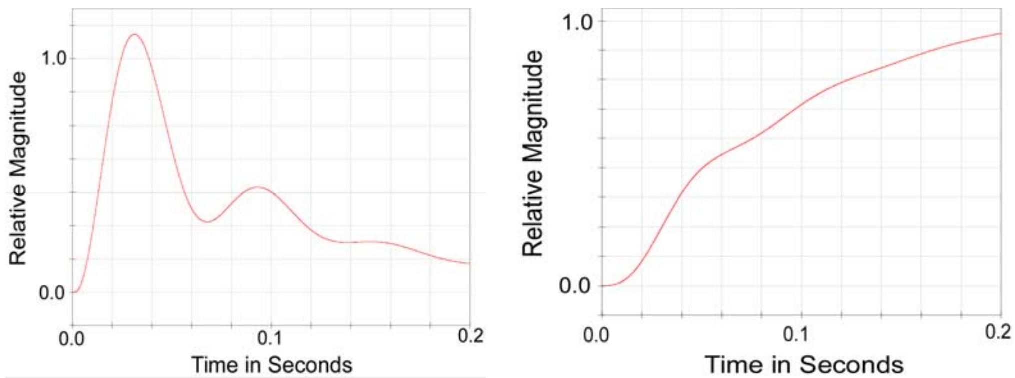

Figure 6.0 – Third-Order Analog Model Impulse and Step Responses

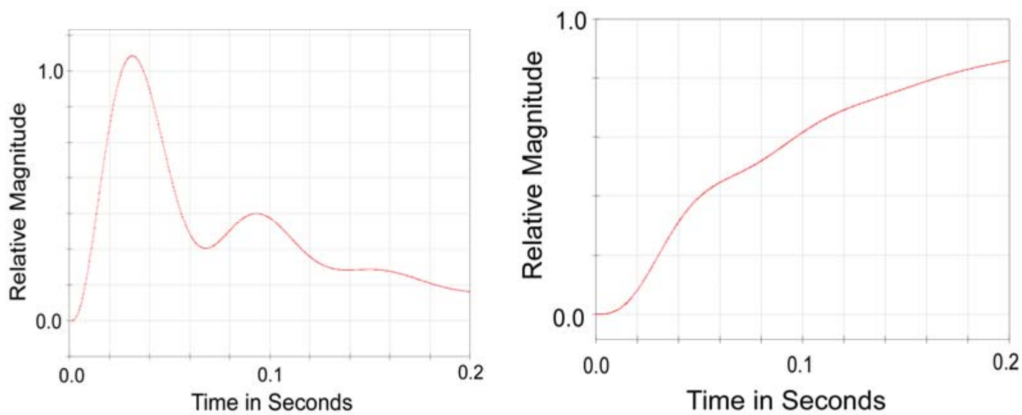

Figure 6.1 – Third-Order Digital Model Impulse and Step Responses

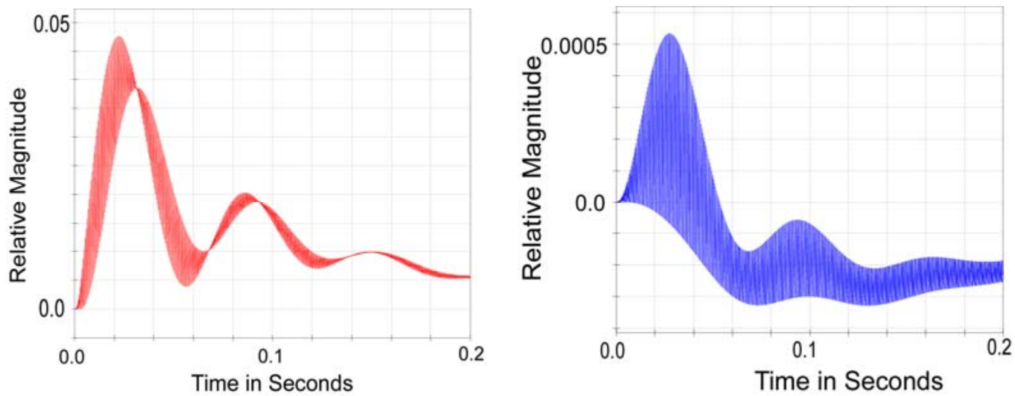

Figure 6.2 – Third-Order Model Impulse and Step Response Differences

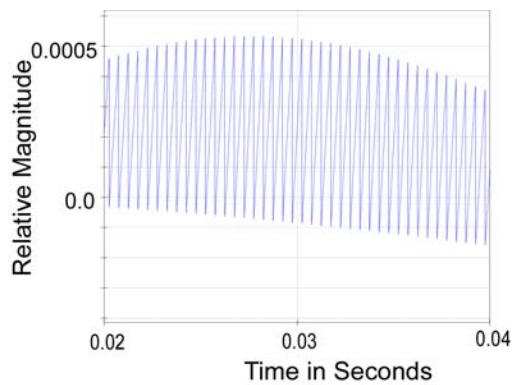

Figure 6.3 – Third-Order Model Expanded Step Response Differences

Figure 6.0 illustrates the analog model behavior and figure 6.1 illustrates the equivalent digital system model for the third-order system. Figure 6.2 illustrates the instantaneous differences in the impulse and step responses. The impulse responses show a peak difference of less than 5% between analog and digital approaches and the peak difference in step responses is ~ 0.5% (an order of magnitude more precise.

A closer examination of an expanded section of the step response difference explains that much of the difference is a "sawtooth" error caused by the digital system matching the analog closely at one point and the error expanding between the continuously varying analog model and the discrete steps of the digital model caused by the ZOH steps.

7.0 Predictor Model Using Shift-Operator Notation–

We have spent a great deal of effort developing a closely matched pair of models, one analog as the reference model and one digital that is intended to match the behavior of the analog model, at least in the time domain.

We now utilize the third-order digital model to construct a predictor of model behavior. We will use the digital predictor to produce an estimate of the future behavior of the analog model. We will then use the predictor to generate a control sequence to control future behavior of the analog model. We will be using the analog model as the ***H(f)*** function that is to be controlled.

We return to the origins of the digital model in equation [6.12], reproduced here for reference:

$$q^3Y(q) = -\left[(a_1 - a_{11})q^2 + (a_2 - a_{11}a_1)q - a_{11}a_2\right]Y(q) = (1 - a_{11})\left[b_1q + b_2\right]X(q) \qquad [6.12]$$

We reformulate equation [6.12] to make the time steps more apparent in the following:

$$\begin{aligned} q^3Y(q) = &(a_{11} - a_1)q^2Y(q) + (a_{11}a_1 - a_2)qY(q) + a_{11}a_2Y(q) \\ &+ (1 - a_{11})b_1qX(q) + (1 - a_{11})b_2X(q) \end{aligned} \qquad [7.0]$$

Equation [7.0] clearly shows that to obtain the value of $q^3Y(q)$, we would need the values of $q^2Y(q)$, $qY(q)$, $Y(q)$, $qX(q)$, and $X(q)$, or at least an estimate of those values. The only values that we have from measurements are $Y(q)$ and $X(q)$ alone. For the development of the digital model, however, we applied the shift operator q to equation [6.12] to obtain the digital estimate of the present value of $Y(q)$ from past and present values of $Y(q)$ and $X(q)$ alone. We have shown that the digital model is a very good representation of the analog system behavior and therefore an accurate model.

We use equation [6.12] as reformulated in equation [7.0] to provide predictions of $q^2Y(q)$ and $qY(q)$, knowing that we can get present and prior estimates of $Y(q)$ from measurements of the analog system.

Before we delve deeply into the algebra that follows, we define new parameters for equation [7.0] to help the manipulations as follows:

$$\begin{aligned} q^3Y(q) = &g_0q^2Y(q) + g_1qY(q) + g_2Y(q) \\ &+ k_1qX(q) + k_2X(q) \end{aligned} \qquad [7.1]$$

$$g_0 = a_{11} - a_1 \qquad [7.2]$$

$$g_1 = a_{11}a_1 - a_2 \qquad [7.3]$$

$$g_2 = a_{11}a_2 \qquad [7.4]$$

$$k_1 = (1 - a_{11})b_1 \qquad [7.5]$$

$$k_2 = (1 - a_{11})b_2 \qquad [7.6]$$

We apply the shift operator ***q*** once to equation [7.1] to obtain the digital estimate of ***q^2Y(q)*** as follows:

$$q^2Y(q) = g_0qY(q) + g_1Y(q) + g_2q^{-1}Y(q) + k_1X(q) + k_2q^{-1}X(q) \qquad [7.7]$$

And we apply the shift operator ***q*** once to equation [7.1] to obtain the digital estimate of ***qY(q)*** as follows:

$$qY(q) = g_0Y(q) + g_1q^{-1}Y(q) + g_2q^{-2}Y(q) + k_1q^{-1}X(q) + k_2q^{-2}X(q) \qquad [7.8]$$

Because all the samples we use in equation [7.8] for ***qY(q)*** prediction have already all occurred, we have all the information we require to make such a “one-step ahead” prediction.

We revisit equation [7.7] for a model of “two-step-ahead” prediction and note that we require a value for ***qY(q)*** in the formulation. That value is available from equation [7.8] and can be substituted with the following result:

$$\begin{aligned} q^2Y(q) = {} & g_0\left[g_0Y(q) + g_1q^{-1}Y(q) + g_2q^{-2}Y(q) + k_1q^{-1}X(q) + k_2q^{-2}X(q)\right] \\ & + g_1Y(q) + g_2q^{-1}Y(q) + k_1X(q) + k_2q^{-1}X(q) \end{aligned} \qquad [7.9]$$

We regroup terms for clarity as follows:

$$\begin{aligned} q^2Y(q) = {} & (g_0^2 + g_1)Y(q) + (g_0g_1 + g_2)q^{-1}Y(q) + g_0g_2q^{-2}Y(q) \\ & + k_1X(q) + (g_0k_1 + k_2)q^{-1}X(q) + g_0k_2q^{-2}X(q) \end{aligned} \qquad [7.10]$$

Because all the samples we use in equation [7.10] for ***q^2Y(q)*** prediction have already all occurred, we have all the information we require to make such a “two-step ahead” prediction.

We have explicit formulas for two future values of ***Y(q)*** for the third-order system in equations [7.8], and [7.10], respectively.

To employ the predictor in a controller, we will need predictions further past the “two-step ahead” prediction. For our example third-order system, we will require predictions for two more steps ahead but defer the development until they are needed.

We could extend the prediction procedure as needed for higher-order systems.

8.0 Control Objective Using the Predictor Model–

We introduce a controller to our system as follows:

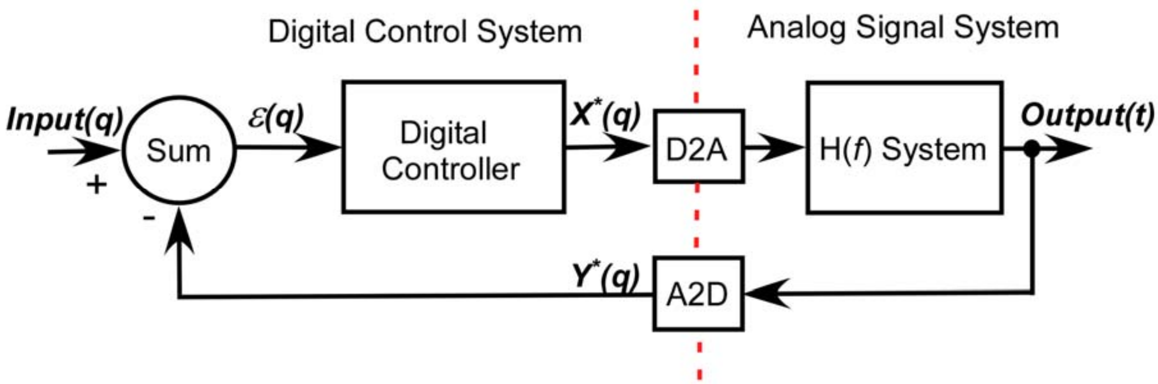

Figure 8.0 – Digital Controller System

We assume that the "Input" signal defines the objective for the system. "Outputs," both in analog and digital form are shown but we are measuring in digital form and have that available in the form of ***Y*(q)***. The Block marked as "Sum" actually provides the difference between the present ***Y*(q)*** and the objective function "***Input(q)***," providing that difference as ***ε(q)***.

The "Digital Controller" Block uses the information in the ***ε(q)*** sequence to provide the sequence ***X*(q)*** to the "Analog to Digital Converter" with the purpose of reducing the error to zero. We will formulate the error sequence ***ε(q)*** itself, and we will keep a history of the prior system excitations ***X*(q)***, and the prior system responses ***Y*(q)***, as well.

In addition, we have analog model equations that we have developed for the sequences ***X(q)*** and ***Y(q)***, and will use those to represent the ***H(f)*** system itself in the spreadsheet form. We will illustrate also, a simulation of the digital controller operating to control the analog simulation, but that cannot easily be represented by a spreadsheet.

Using equation [7.10], we can predict a value for the model output ***Y(q)*** two steps in the future in terms of prior measured ***Y*(q)*** outputs and two of the three prior measured ***X*(q)*** input pulse values, as well as present and future input values we choose to supply to ***H(f)*** through the Digital to analog Converter.

Using the predictor approach, we estimate an error sequence and provide control inputs to reduce the future error and its derivatives to zero. For the third-order system, we set the error and the first and second derivatives to zero. Higher order systems have more derivatives and a longer error sequence needs to be predicted.

We can do little about the present and past error, but we can apply a future sequence of actual inputs to ***X*(q)*** so that the predicted output of the model $q^2Y(q)$ is precisely at the value to control $\varepsilon(q)$ two steps in the future. This strategy is the basis of the "deadbeat" control methodology.

9.0 Control Approach Using the Predictor Model–
We are unable to change past values of the samples. Also, we know that the present value of ***Y*(q)*** was determined by the old values of ***Y*(q)*** and past values of ***X*(q)*** that we applied. From equation [6.14], we know that the best we can do is apply a sequence of ***X*(q)*** values but have no effect on ***Y*(q)*** until two samples in the future.

We do, however, have suitable predictors for ***Y*(q)*** from our digital model that we can employ to estimate expected values for ***qY*(q)*** and $q^2Y^*(q)$ that we can employ for control purposes.

Our approach is to apply control inputs to the system in a sequence of control pulses that we define as a control sequence ***U(q)*** for purposes of calculation as a function of the error sequence $\varepsilon(q)$. Once we calculate the desired sequence, we will provide that sequence as ***X*(q)*** to the Digital to Analog Converter for presentation to the ***H(f)*** analog system.

$$qY_p(q) = g_0Y^*(q) + g_1q^{-1}Y^*(q) + g_2q^{-2}Y^*(q) + k_1q^{-1}X^*(q) + k_2q^{-2}X^*(q) \qquad [9.0]$$

The use of $Y_p(q)$ defines the model prediction value to distinguish it from a measured value. The ***Y*(q)*** and ***X*(q)*** are used to indicate measured values from the system and ***U(q)*** will be used to describe the control sequence that we will compute.

Because we have no better information, we will assume that the ***Input(q)*** remains constant indefinitely. We measure the present value of error as:

$$\varepsilon(q) = Input(q) - Y^*(q) \qquad [9.1]$$

We predict the next-step value of error as:

$$q\varepsilon(q) = Input(q) - qY_p(q) \qquad [9.2]$$

We predict the second-step value of error as:

$$q^2\varepsilon(q) = Input(q) - q^2Y_p(q) \qquad [9.3]$$

One control objective is to set the second-step value of error to zero and two of its derivatives also to zero at that time.

$$q^2\varepsilon(q) = Input(q) - q^2 Y_p(q) = 0 \qquad [9.4]$$

$$q^2 Y_p(q) = Input(q) \qquad [9.5]$$

We represent the first derivative of $\boldsymbol{\varepsilon}$ as $\boldsymbol{\Delta\varepsilon}$ at the second-step time by the first forward difference, and likewise define its value to be zero as follows:

$$\Delta q^2\varepsilon(q) = q^3\varepsilon(q) - q^2\varepsilon(q) = 0 \qquad [9.6]$$

$$\Delta q^2\varepsilon(q) = \left[Input(q) - q^3 Y_p(q)\right] - \left[Input(q) - q^2 Y_p(q)\right] = 0 \qquad [9.7]$$

$$q^2 Y_p(q) - q^3 Y_p(q) = 0 \qquad [9.8]$$

$$q^3 Y_p(q) = q^2 Y_p(q) \qquad [9.9]$$

As equation [9.9] indicates, the values of the "two-step ahead" prediction and a new "three-step ahead" prediction are set equal to ensure that the energy within the model does not change the output.

Similarly, we represent the second derivative $\boldsymbol{\Delta^2\varepsilon}$ at the "two-step ahead" prediction time by the second forward difference using a new "four-step ahead" prediction and also define its value with constant ***Input(q)*** to be zero as follows:

$$\Delta^2 q^2\varepsilon(q) = \left[q^4\varepsilon(q) - q^3\varepsilon(q)\right] - \left[q^3\varepsilon(q) - q^2\varepsilon(q)\right] = 0 \qquad [9.10]$$

$$\left[q^3 Y_p(q) - q^4 Y_p(q)\right] - \left[q^2 Y_p(q) - q^3 Y_p(q)\right] = 0 \qquad [9.11]$$

$$\left[q^3 Y_p(q) - q^4 Y_p(q)\right] = \left[q^2 Y_p(q) - q^3 Y_p(q)\right] \qquad [9.12]$$

From the prior result in equation [9.8], we know:

$$\left[q^3 Y_p(q) - q^4 Y_p(q)\right] = \left[q^2 Y_p(q) - q^3 Y_p(q)\right] = 0 \qquad [9.13]$$

$$q^4 Y_p(q) = q^3 Y_p(q) \qquad [9.14]$$

As equation [9.14] indicates, the values of the "three-step ahead" prediction and a new "four -step ahead" prediction are set equal to ensure that the energy within the model does not change the output.

Our objective then, is to set the error to zero at some time (two steps ahead), and have it remain zero for three steps (equal to the order of the system), to ensure that the system has come to equilibrium at the desired ***Y*(q)*** that is equal to the present ***Input(q)*** value.

10.0 Control Sequence Using the Predictor Model–
Using only past and present information, we have a prediction of the error sequence for three error values in the future. Our control objective is to set the final error to zero and two of its derivatives also to zero at that time. We designate that time to be represented by the error in the sequence at the "two-step ahead" time. To achieve the result, we set:

$$\begin{aligned} q^2Y_p(q) = Input(q) &= \left(g_0^2+g_1\right)Y^*(q)+\left(g_0g_1+g_2\right)q^{-1}Y^*(q)+g_0g_2q^{-2}Y^*(q) \\ &+k_1X^*(q)+\left(g_0k_1+k_2\right)q^{-1}X^*(q)+g_0k_2q^{-2}X^*(q) \end{aligned} \qquad [10.0]$$

To satisfy equation [10.0], we supply the present control input designated by ***X*(q) = U₀***, a value we have not yet determined. Once we have determined a value for U_0, we will supply it to the system as the present value for ***X*(q)*** in the input to ***H(f)***. We calculate U_0 and successive values before we take our next sample and apply three ***U*** values at each appropriate time instant in place of the ***X*(q)*** sequence, whereupon they become ***X*(q)*** values.

We employ equation [10.0] to prepare a prediction for the next step to the "three-step ahead" time as follows:

$$\begin{aligned} q^3Y_p(q) &= \left(g_0^2+g_1\right)qY_p(q)+\left(g_0g_1+g_2\right)Y^*(q)+g_0g_2q^{-1}Y^*(q) \\ &+k_1qX^*(q)+\left(g_0k_1+k_2\right)X^*(q)+g_0k_2q^{-1}X^*(q) \end{aligned} \qquad [10.1]$$

We note that the "three-step ahead" prediction refers to the present ***X*(q)*** value, that we have already decided will be U_0, and in addition, a future ***qX*(q)*** input that we will designate as U_1 to be computed in the following sections. We continue using the ***X*(q)*** notation to avoid confusion with the appropriate delays or prediction timing. Recall though that present and future values of ***X*(q)*** will be computed as control ***U*** and substituted at the appropriate sequence times.

We employ equation [9.0], to prepare the value for the "one-step ahead prediction" required for substitution as follows:

$$\begin{aligned} q^3Y_p(q) &= \left(g_0^2+g_1\right)\left[g_0Y^*(q)+g_1q^{-1}Y^*(q)+g_2q^{-2}Y^*(q)+k_1q^{-1}X^*(q)+k_2q^{-2}X^*(q)\right] \\ &+\left(g_0g_1+g_2\right)Y^*(q)+g_0g_2q^{-1}Y^*(q) \end{aligned}$$

$$+k_1 q X^*(q) + (g_0 k_1 + k_2) X^*(q) + g_0 k_2 q^{-1} X^*(q) \qquad [10.2]$$

We collect terms to provide an explicit prediction form as follows:

$$q^3Y_p(q)=\left(g_0^3+2g_0g_1+g_2\right)Y^*(q)+\left(g_1g_0^2+g_1^2+g_0g_2\right)q^{-1}Y^*(q)+\left(g_0^2+g_1\right)g_2q^{-2}Y^*(q)$$
$$+k_1qX^*(q)+\left(g_0k_1+k_2\right)X^*(q)+\left(g_0k_2+g_0^2k_1+g_1k_1\right)q^{-1}X^*(q)+\left(g_0^2+g_1\right)k_2q^{-2}X^*(q) \quad [10.3]$$

We employ equation [9.3] to prepare a prediction for the next step to the "four-step ahead" time in preparation for substitution of equation [9.0] for the requisite "one-step ahead" prediction as follows:

$$\begin{aligned}q^4Y_p(q)=&\left(g_0^3+2g_0g_1+g_2\right)qY^*(q)\\&+\left(g_1g_0^2+g_1^2+g_0g_2\right)Y^*(q)\\&+\left(g_0^2+g_1\right)g_2q^{-1}Y^*(q)\\&+k_1q^2X^*(q)+\left(g_0k_1+k_2\right)qX^*(q)+\left(g_0k_2+g_0^2k_1+g_1k_1\right)X^*(q)\\&+\left(g_0^2+g_1\right)k_2q^{-1}X^*(q)\end{aligned} \quad [10.4]$$

With the substitution of equation [9.0] for the requisite "one-step ahead" prediction:

$$\begin{aligned}q^4Y_p(q)=&\left(g_0^3+2g_0g_1+g_2\right)\left[g_0Y^*(q)+g_1q^{-1}Y^*(q)+g_2q^{-2}Y^*(q)+k_1q^{-1}X^*(q)+k_2q^{-2}X^*(q)\right]\\&+\left(g_1g_0^2+g_1^2+g_0g_2\right)Y^*(q)\\&+\left(g_0^2+g_1\right)g_2q^{-1}Y^*(q)\\&+k_1q^2X^*(q)+\left(g_0k_1+k_2\right)qX^*(q)+\left(g_0k_2+g_0^2k_1+g_1k_1\right)X^*(q)\\&+\left(g_0^2+g_1\right)k_2q^{-1}X^*(q)\end{aligned} \quad [10.5]$$

We expand the "one-step ahead" prediction as follows:

$$\begin{aligned}q^4Y_p(q)=&\left(g_0^3+2g_0g_1+g_2\right)g_0Y^*(q)+\left(g_0^3+2g_0g_1+g_2\right)g_1q^{-1}Y^*(q)+\left(g_0^3+2g_0g_1+g_2\right)g_2q^{-2}Y^*(q)\\&+\left[\left(g_0^3+2g_0g_1+g_2\right)k_1q^{-1}X^*(q)+\left(g_0^3+2g_0g_1+g_2\right)k_2q^{-2}X^*(q)\right]\\&+\left(g_1g_0^2+g_1^2+g_0g_2\right)Y^*(q)\\&+\left(g_0^2+g_1\right)g_2q^{-1}Y^*(q)\\&+k_1q^2X^*(q)+\left(g_0k_1+k_2\right)qX^*(q)+\left(g_0k_2+g_0^2k_1+g_1k_1\right)X^*(q)\\&+\left(g_0^2+g_1\right)k_2q^{-1}X^*(q)\end{aligned} \quad [10.6]$$

We expand the result as follows:

$$\begin{aligned}q^4Y_p(q)=&\left(g_0^3+2g_0g_1+g_2\right)g_0Y^*(q)+\left(g_0^3+2g_0g_1+g_2\right)g_1q^{-1}Y^*(q)+\left(g_0^3+2g_0g_1+g_2\right)g_2q^{-2}Y^*(q)\\&+\left[\left(g_0^3+2g_0g_1+g_2\right)k_1q^{-1}X^*(q)+\left(g_0^3+2g_0g_1+g_2\right)k_2q^{-2}X^*(q)\right]\\&+\left(g_1g_0^2+g_1^2+g_0g_2\right)Y^*(q)\\&+\left(g_0^2+g_1\right)g_2q^{-1}Y^*(q)\\&+k_1q^2X^*(q)+\left(g_0k_1+k_2\right)qX^*(q)+\left(g_0k_2+g_0^2k_1+g_1k_1\right)X^*(q)\\&+\left(g_0^2+g_1\right)k_2q^{-1}X^*(q)\end{aligned} \quad [10.7]$$

We collect similar terms as follows:

$$\begin{aligned}
q^4Y_p(q) = &\left[\left(g_0^3+2g_0g_1+g_2\right)g_0+g_1g_0^2+g_1^2+g_0g_2\right]Y^*(q) \\
&+\left[\left(g_0^3+2g_0g_1+g_2\right)g_1+\left(g_0^2+g_1\right)g_2\right]q^{-1}Y^*(q) \\
&+\left(g_0^3+2g_0g_1+g_2\right)g_2q^{-2}Y^*(q) \\
&+\left[\left(g_0^3+2g_0g_1+g_2\right)k_1+\left(g_0^2+g_1\right)k_2\right]q^{-1}X^*(q) \\
&+\left(g_0^3+2g_0g_1+g_2\right)k_2q^{-2}X^*(q) \\
&+k_1q^2X^*(q)+\left(g_0k_1+k_2\right)qX^*(q)+\left(g_0k_2+g_0^2k_1+g_1k_1\right)X^*(q)
\end{aligned} \qquad [10.8]$$

Equations [10.0], [10.3], and [10.8] for the two-step, three-step, and four-step ahead predictions represent three simultaneous equations in three unknown ***U*** values, allowing for the explicit solution of those values. Each prediction is set to the value of the Input and solved simultaneously.

We express the equations in somewhat different form to emphasize the role of the control sequence as follows:

$$\begin{aligned}
Input(q)-&\left[\left(g_0^2+g_1\right)Y^*(q)+\left(g_0g_1+g_2\right)q^{-1}Y^*(q)+g_0g_2q^{-2}Y^*(q)\right. \\
&\left.+\left(g_0k_1+k_2\right)q^{-1}X^*(q)+g_0k_2q^{-2}X^*(q)\right]=k_1X^*(q)
\end{aligned} \qquad [10.9]$$

$$\begin{aligned}
Input(q)-&\left[\left(g_0^3+2g_0g_1+g_2\right)Y^*(q)+\left(g_1g_0^2+g_1^2+g_0g_2\right)q^{-1}Y^*(q)+\left(g_0^2+g_1\right)g_2q^{-2}Y^*(q)\right. \\
&\left.+\left(g_0k_2+g_0^2k_1+g_1k_1\right)q^{-1}X^*(q)+\left(g_0^2+g_1\right)k_2q^{-2}X^*(q)\right]=k_1qX^*(q)+\left(g_0k_1+k_2\right)X^*(q)
\end{aligned} \qquad [10.3]$$

$$\begin{aligned}
Input(q)-&\left[\left[\left(g_0^3+2g_0g_1+g_2\right)g_0+g_1g_0^2+g_1^2+g_0g_2\right]Y^*(q)\right. \\
&+\left[\left(g_0^3+2g_0g_1+g_2\right)g_1+\left(g_0^2+g_1\right)g_2\right]q^{-1}Y^*(q) \\
&+\left(g_0^3+2g_0g_1+g_2\right)g_2q^{-2}Y^*(q) \\
&+\left[\left(g_0^3+2g_0g_1+g_2\right)k_1+\left(g_0^2+g_1\right)k_2\right]q^{-1}X^*(q) \\
&\left.+\left(g_0^3+2g_0g_1+g_2\right)k_2q^{-2}X^*(q)\right] \\
&=k_1q^2X^*(q)+\left(g_0k_1+k_2\right)qX^*(q)+\left(g_0k_2+g_0^2k_1+g_1k_1\right)X^*(q)
\end{aligned} \qquad [10.8]$$

We are faced with solving the tree simultaneous equations, but the dependence of factors on the sequence of ***Y*(q)*** and ***X*(q)*** sequence values hides the simplicity of the problem somewhat. From these three equations [10.9], [10.3], and [10.8], we define three variables ***C_0(q)***, ***C_1(q)***, and ***C_2(q)*** as follows:

$$\begin{aligned}
C_0(q)=&\left(g_0^2+g_1\right)Y^*(q)+\left(g_0g_1+g_2\right)q^{-1}Y^*(q)+g_0g_2q^{-2}Y^*(q) \\
&+\left(g_0k_1+k_2\right)q^{-1}X^*(q)+g_0k_2q^{-2}X^*(q)
\end{aligned} \qquad [10.9]$$

$$\begin{aligned}
C_1(q)=&\left(g_0^3+2g_0g_1+g_2\right)Y^*(q)+\left(g_1g_0^2+g_1^2+g_0g_2\right)q^{-1}Y^*(q)+\left(g_0^2+g_1\right)g_2q^{-2}Y^*(q) \\
&+\left(g_0k_2+g_0^2k_1+g_1k_1\right)q^{-1}X^*(q)+\left(g_0^2+g_1\right)k_2q^{-2}X^*(q)
\end{aligned} \qquad [10.10]$$

$$\begin{aligned}C_2(q) = &\left[\left(g_0^3 + 2g_0g_1 + g_2\right)g_0 + g_1g_0^2 + g_1^2 + g_0g_2\right]Y^*(q)\\ &+\left[\left(g_0^3 + 2g_0g_1 + g_2\right)g_1 + \left(g_0^2 + g_1\right)g_2\right]q^{-1}Y^*(q)\\ &+\left(g_0^3 + 2g_0g_1 + g_2\right)g_2q^{-2}Y^*(q)\\ &+\left[\left(g_0^3 + 2g_0g_1 + g_2\right)k_1 + \left(g_0^2 + g_1\right)k_2\right]q^{-1}X^*(q)\\ &+\left(g_0^3 + 2g_0g_1 + g_2\right)k_2q^{-2}X^*(q)\end{aligned} \qquad [10.11]$$

Using equation [10.9], [10.10], and [10.11] definitions, we rewrite the simultaneous equations [10.9], [10.3], and [10.8] as follows:

$$Input(q) - C_0(q) = k_1X^*(q) \qquad [10.12]$$

$$Input(q) - C_1(q) = (g_0k_1 + k_2)X^*(q) + k_1qX^*(q) \qquad [10.13]$$

$$Input(q) - C_2(q) = \left(g_0k_2 + g_0^2k_1 + g_1k_1\right)X^*(q) + (g_0k_1 + k_2)qX^*(q) + k_1q^2X^*(q) \qquad [10.14]$$

We substitute the pulse values $\boldsymbol{U_0}$, $\boldsymbol{U_1}$, and $\boldsymbol{U_2}$ as follows:

$$Input(q) - C_0(q) = k_1U_0 \qquad [10.15]$$

$$Input(q) - C_1(q) = (g_0k_1 + k_2)U_0 + k_1U_1 \qquad [10.16]$$

$$Input(q) - C_2(q) = \left(g_0k_2 + g_0^2k_1 + g_1k_1\right)U_0 + (g_0k_1 + k_2)U_1 + k_1U_2 \qquad [10.17]$$

In Table 9.0 below, the same third-order digital model is illustrated with a step input and outputs calculated as $\boldsymbol{Y(n)}$. At each value of "n," The coefficients $\boldsymbol{C_0(q)}$, $\boldsymbol{C_1(q)}$, and $\boldsymbol{C_2(q)}$ are calculated. Each coefficient is used, to calculate the predictors $\boldsymbol{q^2Y_p(q)}$, $\boldsymbol{q^3Y_p(q)}$, and $\boldsymbol{q^4Y_p(q)}$. Each predictor value is used to provide the difference between the prediction value and the model value at the appropriate value of "n." The Root-Sum-Square (RSS) of the predictor errors is provided in the RSS column. At each value of "n," the RSS error is smaller than 1 part in 10 digits, and verifies that the calculations for the $\boldsymbol{C_0(q)}$, $\boldsymbol{C_1(q)}$, and $\boldsymbol{C_2(q)}$ as well as $\boldsymbol{q^2Y_p(q)}$, $\boldsymbol{q^3Y_p(q)}$, and $\boldsymbol{q^4Y_p(q)}$ are correct. Only the first 10 of 1000 rows are shown, but the results are consistent.

F_1	F_2	ζ	π	τ	T_S	T_S
1.6	16	0.2500	3.14E+00	1.00E-02	1.00E-04	1.00E-04

Block Diagram Model Coefficients

a_1	-0.603186	g_0	2.98E+00
a_2	-1.061180	g_1	-2.99E+00
a_3	-0.101601	g_2	9.94E-01
b_3	0.101601		
		k_1	7.60E-06

Shift Operator Model Coefficients

		k_2	-7.40E-06
ω_0	1.01E+02		
α	9.90E-01		
ω	9.73E+01		
β	1.00E+00		
γ	9.73E-03	a_{12}	1.00E-03

First-Order Model Coefficient

a_{11}	9.99E-01

Second-Order Model Coefficient

a_1	-1.98E+00
a_2	9.80E-01
b_1	7.56E-03
b_2	-7.37E-03

						RSS	Predictors		
Input	**n**	**Y(n)**	**C_0**	**C_1**	**C_2**	**Pred error**	**$q^2Yp(n)$**	**$q^3Yp(n)$**	**$q^4Yp(n)$**
0	0	0.0000E+00							
0	1	0.0000E+00							
1	2	0.0000E+00	0.00E+00	0.00E+00	0.00E+00	1.40E-20	7.60E-06	2.28E-05	4.57E-05
1	3	0.0000E+00	1.52E-05	2.29E-05	3.06E-05	3.05E-20	2.28E-05	4.57E-05	7.63E-05
1	4	7.5979E-06	3.81E-05	5.35E-05	6.89E-05	6.21E-20	4.57E-05	7.63E-05	1.15E-04
1	5	2.2828E-05	6.87E-05	9.18E-05	1.15E-04	3.03E-20	7.63E-05	1.15E-04	1.61E-04
1	6	4.5723E-05	1.07E-04	1.38E-04	1.69E-04	4.07E-20	1.15E-04	1.61E-04	2.15E-04
1	7	7.6311E-05	1.53E-04	1.92E-04	2.30E-04	2.94E-19	1.61E-04	2.15E-04	2.76E-04
1	8	1.1462E-04	2.07E-04	2.53E-04	3.00E-04	4.07E-19	2.15E-04	2.76E-04	3.46E-04
1	9	1.6068E-04	2.69E-04	3.23E-04	3.77E-04	9.42E-19	2.76E-04	3.46E-04	4.23E-04
1	10	2.1451E-04	3.38E-04	4.00E-04	4.62E-04	1.77E-18	3.46E-04	4.23E-04	5.08E-04

Table 10.0 Predictor Validation Calculations

11.0 Control Sequence Calculation Using the Predictor Model–
We solve the simultaneous equations [10.9], [10.10], and [10.11] for the control sequence by inspection as follows:

$$U_0 = \frac{[Input(q) - C_0(q)]}{k_1} \qquad [11.0]$$

$$U_1 = \frac{[Input(q) - C_1(q) - (g_0k_1 + k_2)U_0]}{k_1} \qquad [11.1]$$

$$U_1 = \frac{[Input(q) - C_2(q) - (g_0k_2 + g_0^2k_1 + g_1k_1)U_0 - (g_0k_1 + k_2)U_1]}{k_1} \qquad [11.2]$$

In	n	C0	C1	C2	U0	U1	U2	X* (n)	Y* (n)
0	0							0	
0	1							0	
1	2							0	0.00E+00
1	3							0	0.00E+00
1	4	3.81E-05	5.35E-05	6.89E-05	1.32E+05	-1.32E+05	4.35E+01	1.32E+05	0.00E+00
1	5							-1.32E+05	0.00E+00
1	6							4.35E+01	1.00E+00
1	7							0	1.00E+00
1	8							0	1.00E+00

Table 11.0 Control Validation Calculations

We use the values obtained from [11.0], [11.1], and [11.2] and produce the listing shown in Table 11.0, but we have not shown the previous ***Y(n)*** that was referenced to produce the $\boldsymbol{C_0(q)}$, $\boldsymbol{C_1(q)}$, and $\boldsymbol{C_2(q)}$. The ***Y(n)*** was used to obtain the values so that we can employ them for control of another model that we list as $\boldsymbol{X^*(n)}$ and $\boldsymbol{Y^*(n)}$ to show the effect of applying the $\boldsymbol{U_0(q)}$, $\boldsymbol{U_1(q)}$, and $\boldsymbol{U_2(q)}$ sequence to that similar system. The original ***Y(n)*** system response is shown in Table 9.0 and it results from a unit-step input at the ***n*** = 2 time. At the ***n*** = 4 instant, we utilize the calculated values of the $\boldsymbol{U_0(q)}$, $\boldsymbol{U_1(q)}$, and $\boldsymbol{U_2(q)}$ sequence to apply them to the $\boldsymbol{Y^*(n)}$ system as $\boldsymbol{X^*(n)}$ inputs. Both the $\boldsymbol{Y^*(n)}$ and $\boldsymbol{X^*(n)}$ sequences have been quiescent prior to the ***n*** = 4 instant. $\boldsymbol{Y^*(n)}$ shows no response for the ***n*** = 4, and 5 instants, but for the ***n*** = 6 instant, $\boldsymbol{Y^*(n)}$ has attained the value of unity.

We have shown that the $\boldsymbol{U_0(q)}$, $\boldsymbol{U_1(q)}$, and $\boldsymbol{U_2(q)}$ sequence is effective in providing the requisite control signal to provide a fast step response. The system is running "openloop" following the $\boldsymbol{n}$ = 6 instant, but the natural response to a $\boldsymbol{X^*(n)}$ = 0 control sequence keeps the $\boldsymbol{Y^*(n)}$ sequence close to the desired $\boldsymbol{Y^*(n)}$ = 1 value.

12.0 Control Sequence Stability –

On examination of equations [9.0], [9.1], and [9.2] we see that the $\boldsymbol{U_0(q)}$, $\boldsymbol{U_1(q)}$, and $\boldsymbol{U_2(q)}$ sequence is determined by measured values of $\boldsymbol{Y^*(q)}$ and past values of the $\boldsymbol{X^*(q)}$ sequence and the ***Input(q)*** signals, as expected, so long as we do not eliminate the dependence on the $\boldsymbol{X^*(q)}$ values..

Another way to look at this equation set is as a system specification for the controller itself, with the sequence of values of ***U*** being the controller output. We have a sequence of three values of ***U*** specified by changing of the ***C(q)***coefficients to provide the sequence ***U(q)***, ***qU(q)***, and $\boldsymbol{q^2U(q)}$ that we seek for control. Because values of ***U*** appear at the controller's output, they are also the $\boldsymbol{X^*(q)}$ sequence.

It is possible to utilize the equation for $\boldsymbol{q^2U(q)}$ and substitute the predicted values for ***U(q)***, ***qU(q)*** as we know they will appear at the $\boldsymbol{X^*(q)}$ sequence controller output, and examine the resulting equation for root locations, however, it is not uncommon that the controller itself is unstable.

In	n	C_0	C_1	C_2	U_0	U_1	U_2	$Y_{input}(n)$	$Y_{analog}(n)$
0	0							0.0000E+00	0.00E+00
0	1							0.0000E+00	0.00E+00
1	2							0.0000E+00	0.00E+00
1	3							0.0000E+00	0.00E+00
1	4	0.00E+00	0.00E+00	0.00E+00	1.59E+05	-4.71E+05	6.21E+05	1.5923E+05	0.00E+00
1	5							-4.7122E+05	0.00E+00
1	6							6.2180E+05	1.00E+00
1	7	4.87E+00	1.24E+01	2.37E+01	-6.16E+05	6.11E+05	-6.06E+05	-6.1661E+05	1.00E+00
1	8							6.1146E+05	1.00E+00
1	9							-6.0636E+05	1.00E+00
1	10	-2.77E+00	-1.02E+01	-2.11E+01	6.01E+05	-5.96E+05	5.91E+05	6.0130E+05	1.00E+00
1	11							-5.9628E+05	1.00E+00
1	12							5.9131E+05	1.00E+00
1	13	4.68E+00	1.19E+01	2.26E+01	-5.86E+05	5.81E+05	-5.76E+05	-5.8638E+05	1.00E+00
1	14							5.8149E+05	1.00E+00
1	15							-5.7663E+05	1.00E+00
1	16	-2.59E+00	-9.65E+00	-2.00E+01	5.71E+05	-5.67E+05	5.62E+05	5.7182E+05	1.00E+00
1	17							-5.6705E+05	1.00E+00
1	18							5.6232E+05	1.00E+00

Table 12.0 Control Application

13.0 Control Sequence Application to the Analog System –

The controller calculations shown in Table 12.0 above are applied to the analog system with the result shown in Figure 13.0, below:

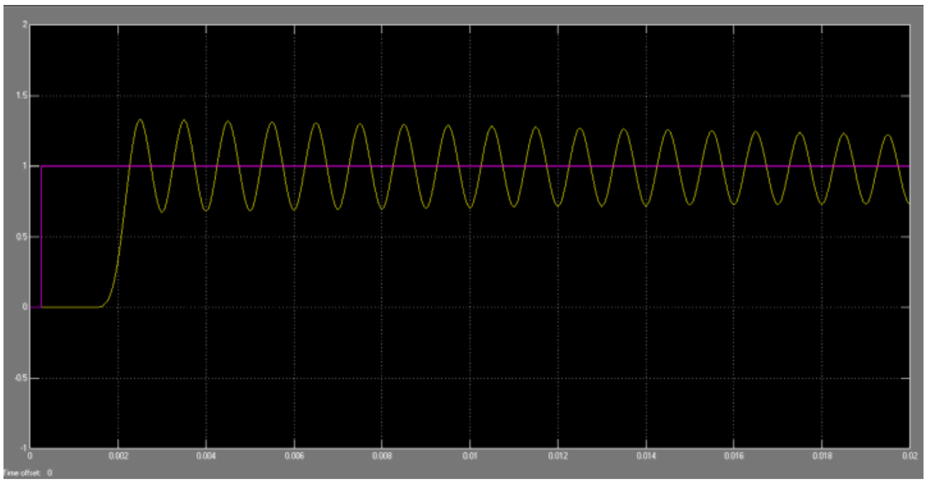

Figure 13.0 Control Application to Analog model

One issue with the simple "deadbeat" control methodology is the discrete-time nature of the control itself. We see that the "Analog System" performs as claimed and the error is zero at the designated sampling instants. Unfortunately, between the sample instants, the system shows a damped sinusoidal response at the Nyquist rate. Because the Analog System itself is stable, eventually, the sinusoid decays to zero on the step waveform.

14.0 Control Sequence Effort –

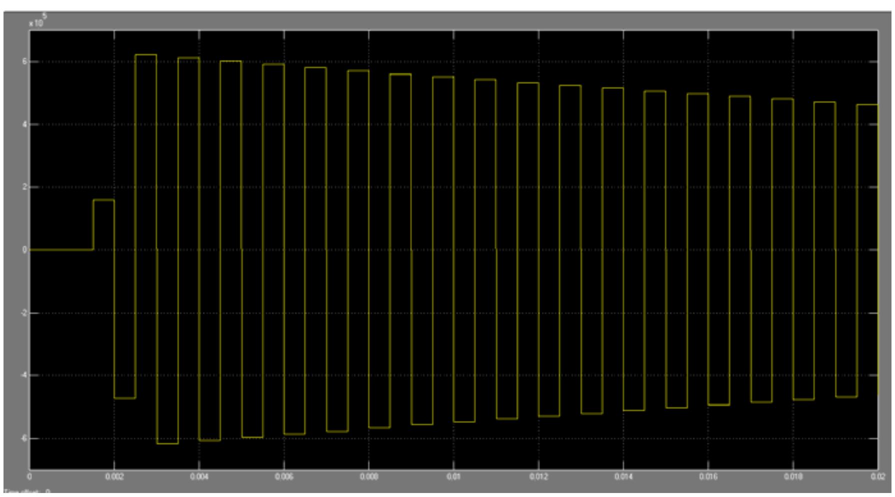

Figure 14.0 Control Applied to Analog model input

The amount of the control signal required to force the system to respond rapidly is also quite high in this example. The situation is similar to forcing a “dump truck” to perform like a “sports car.” It can be done, but the amount of effort is quite high as shown in Figure 14.0 above.

To achieve the rapid system response, the control signal is quite large. For a single unit step response, with a 500 μsec sampling clock, we require control signals of the order of 600,000 units at the input. For some systems this may not present a problem, but for other systems it is a prohibitive cost.

We have utilized a very short sample time to represent our system accurately, but we are not constrained in the same way as we approach the control effort. We can extend the period of the control signal and decrease the control effort required to bring the system to the desired output state.

15.0 Control Sequence Application to the Analog System at 1/10 the Sample Rate– The controller calculations shown in Table 15.0 below, obtained by changing the sampling clock to a 5 msec period are applied to the analog system

In	n	C_0	C_1	C_2	U_0	U_1	U_2	$Y_{input}(n)$	$Y_{analog}(n)$
0	0							0.0000E+00	0.00E+00
0	1							0.0000E+00	0.00E+00
1	2							0.0000E+00	0.00E+00
1	3							0.0000E+00	0.00E+00
1	4	0.00E+00	0.00E+00	0.00E+00	1.78E+02	-4.34E+02	5.33E+02	1.7895E+02	0.00E+00
1	5							-4.3467E+02	0.00E+00
1	6							5.3374E+02	1.00E+00
1	7	3.73E+00	7.84E+00	1.19E+01	-4.88E+02	4.50E+02	-4.12E+02	-4.8860E+02	1.00E+00
1	8							4.5095E+02	1.00E+00
1	9							-4.1251E+02	1.00E+00
1	10	-1.12E+00	-4.35E+00	-7.63E+00	3.81E+02	-3.48E+02	3.21E+02	3.8103E+02	1.00E+00
1	11							-3.4825E+02	1.00E+00
1	12							3.2197E+02	1.00E+00
1	13	2.64E+00	5.11E+00	7.59E+00	-2.93E+02	2.72E+02	-2.48E+02	-2.9398E+02	1.00E+00
1	14							2.7209E+02	1.00E+00
1	15							-2.4814E+02	1.00E+00
1	16	-2.85E-01	-2.23E+00	-4.22E+00	2.29E+02	-2.09E+02	1.94E+02	2.2996E+02	1.00E+00
1	17							-2.0942E+02	1.00E+00
1	18							1.9438E+02	1.00E+00

Table 15.0 Control Application

Slowing the system sampling time by ten times has not prevented the control algorithm from achieving the “deadbeat” response

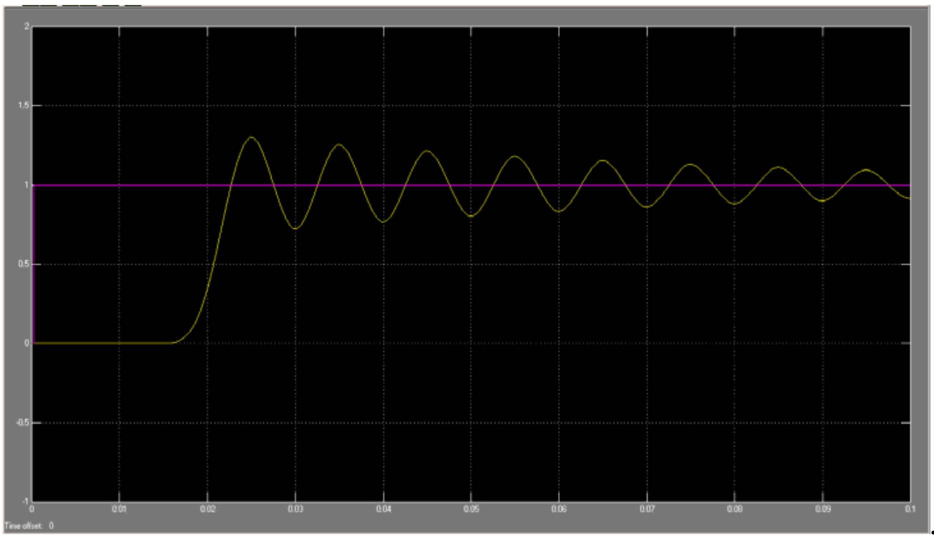

Figure 15.0 Control Application to Analog model

Again, we see the simple “deadbeat” control methodology has achieved the zero error conditions at the sampling instants. Again, between the sample instants, the system shows a damped sinusoidal response at the Nyquist rate. At the slower rate, the damping is greater, and the sinusoid decays more quickly.

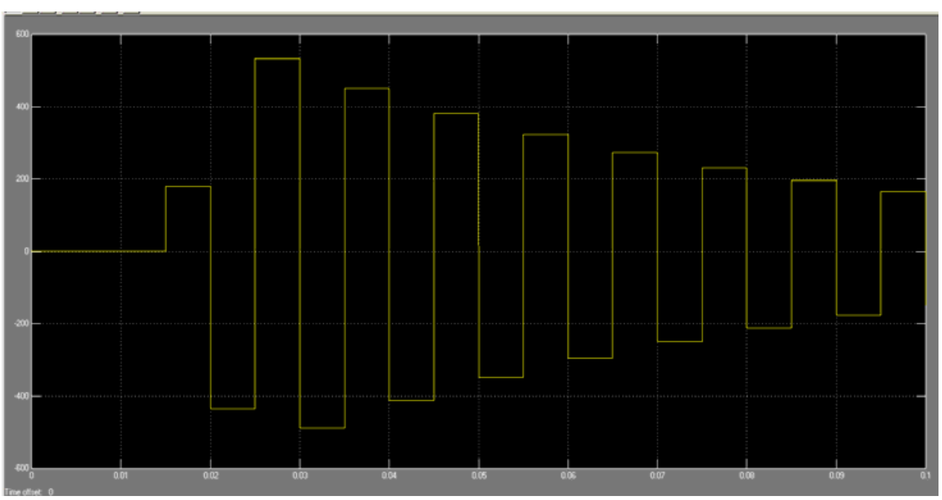

Figure 15.1 Control Applied to Analog model input

The longer sample time still represents our system accurately, but we have greatly reduced the control effort. The extended period of the control signal has decreased the control effort required to bring the system to the desired output state. The peak control effort for the single unit step response, with a 5 msec sampling clock, now requires control signals of the order of 500 units at the input.

16.0 Control Sequence Application at 1/4 the Prior Sample Rate–
The controller calculations shown in Table 16.0 below, obtained by changing the sampling clock to a 20 msec period are applied to the analog system

In	n	C_0	C_1	C_2	U_0	U_1	U_2	$Y_{input}(n)$	$Y_{analog}(n)$
0	0							0.0000E+00	0.00E+00
0	1							0.0000E+00	0.00E+00
1	2							0.0000E+00	0.00E+00
1	3							0.0000E+00	0.00E+00
1	4	0.00E+00	0.00E+00	0.00E+00	5.09E+00	-2.79E-01	3.39E+00	5.0986E+00	0.00E+00
1	5							-2.7854E-01	0.00E+00
1	6							3.3966E+00	1.00E+00
1	7	1.12E+00	7.16E-01	5.64E-01	-6.32E-01	2.11E+00	2.43E-01	-6.3209E-01	1.00E+00
1	8							2.1115E+00	1.00E+00
1	9							2.4309E-01	1.00E+00
1	10	7.02E-01	5.59E-01	5.06E-01	1.51E+00	6.48E-01	1.23E+00	1.5155E+00	1.00E+00
1	11							6.4897E-01	1.00E+00
1	12							1.2390E+00	1.00E+00
1	13	8.35E-01	6.08E-01	5.24E-01	8.37E-01	1.11E+00	9.24E-01	8.3721E-01	1.00E+00
1	14							1.1109E+00	1.00E+00
1	15							9.2450E-01	1.00E+00
1	16	7.93E-01	5.93E-01	5.18E-01	1.05E+00	9.64E-01	1.02E+00	1.0514E+00	1.00E+00
1	17							9.6499E-01	1.00E+00
1	18							1.0238E+00	1.00E+00

Table 16.0 Control Application

Slowing the system sampling time by four more times has not prevented the control algorithm from achieving the “deadbeat” response

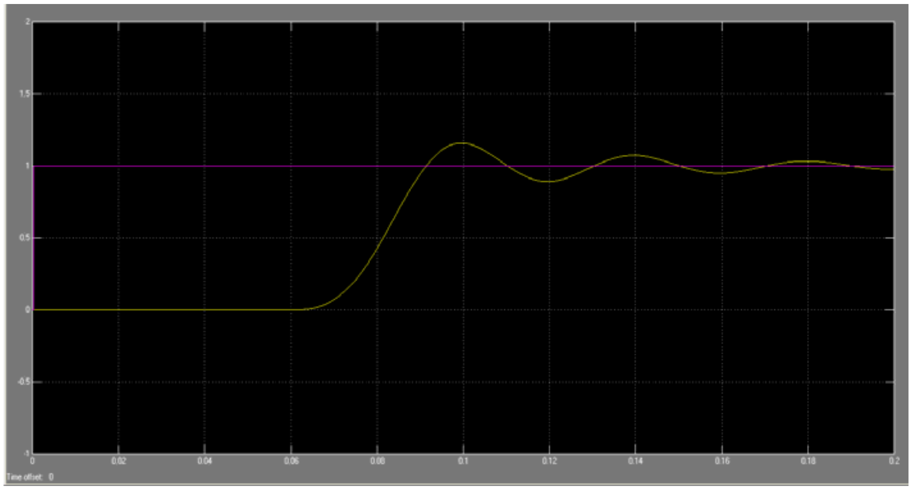

Figure 16.0 Control Application to Analog model

Again, we see the simple "deadbeat" control methodology has achieved the zero error conditions at the sampling instants. Again, between the sample instants, the system shows a damped sinusoidal response at the Nyquist rate. At the slower rate, the damping is greater, and the sinusoid decays more quickly.

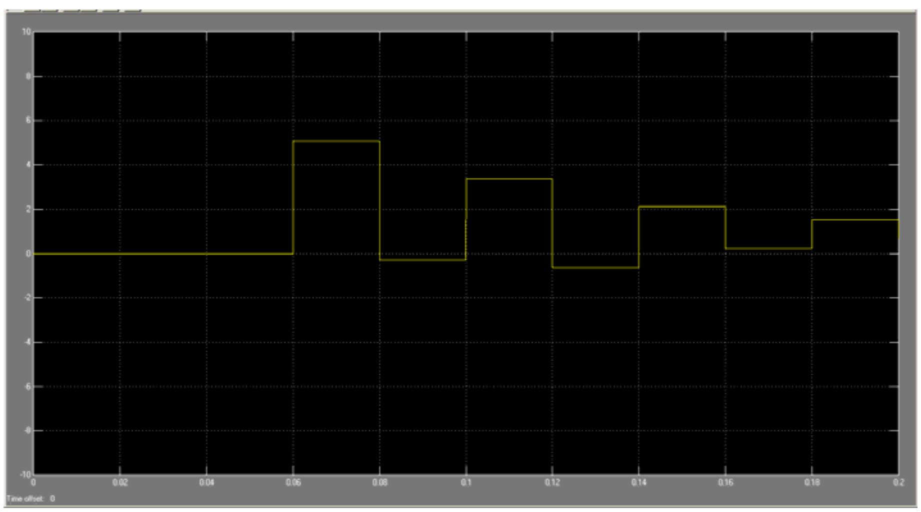

Figure 16.1 Control Applied to Analog model input

The longer sample time still represents our system accurately, but we have greatly reduced the control effort. The extended period of the control signal has decreased the control effort required to bring the system to the desired output state. The peak control effort for the single unit step response, with a 5 msec sampling clock, now requires control signals of less than 5 units at the input.

In Figure 16.0, we also see that the damping of the response is further improved. In contrast, the system has settled near the unit level in 200 msec, whereas the original third-order system had not yet reached 70% of that level in the same time and required nearly a full second to settle to a unit level.

17.0 Control Sequence Application at 20 msec and 50 msec clocking–
The controller was implemented using an analog circuit simulator and discrete-time ZOH analog signal representations to produce the control of the following analog models. The results are similar to the results above and differences are attributable to the less accurate coefficient representations. Other simulations using the analog circuit simulator and discrete-time ZOH analog signal representations exhibited a general lack of control that is expected due to the extreme coefficient sensitivity of the deadbeat methodology. The deadbeat methodology requires the accuracy of the digital computation both in magnitudes and timing for control at the higher sampling rates.

Figures 17.0 and 17.1 illustrate both the digital model and analog model responses with the loop closed around the analog model, with a sampling clock with 20 millisecond period.

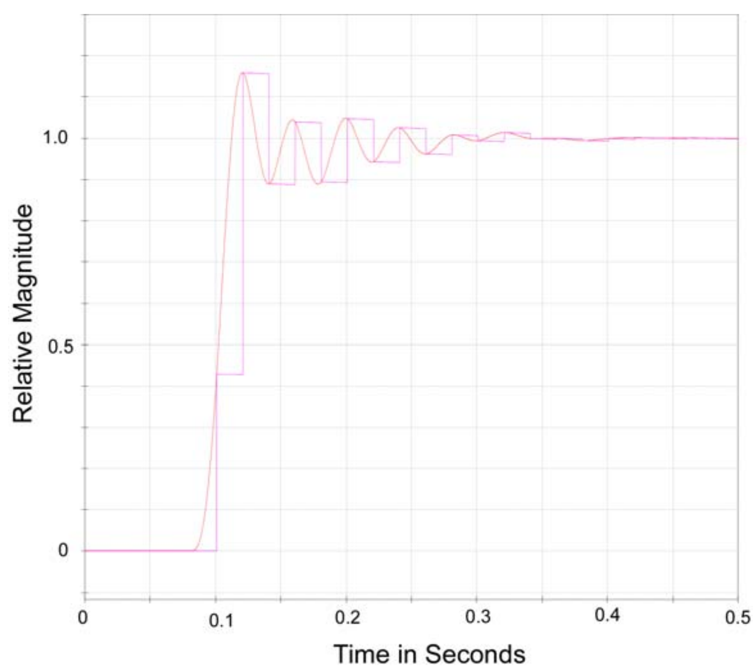

Figure 17.0 Control Applied to Analog model input and Model Response

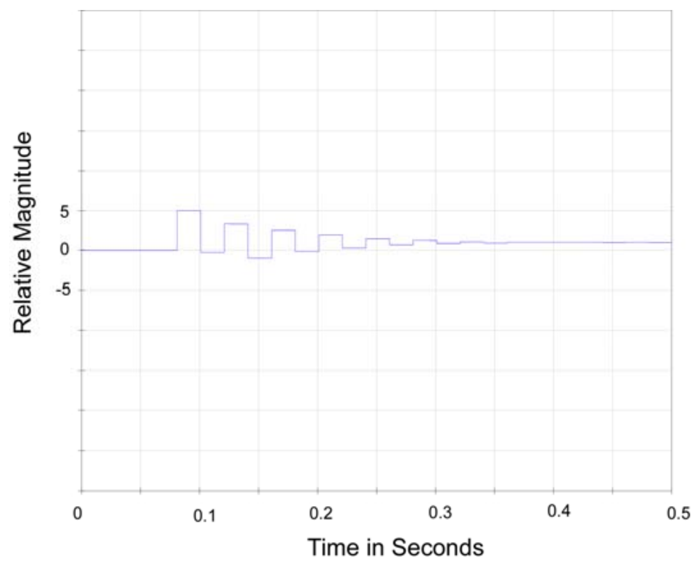

Figure 17.1 Control Signal Applied to Analog model input

Figures 17.2 and 17.3 illustrate both the digital model and analog model responses with the loop closed around the analog model, with a sampling clock with 50 millisecond period.

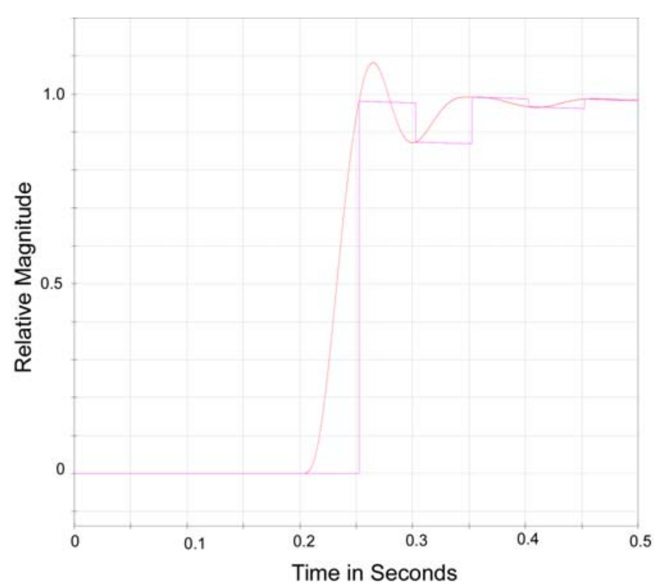

Figure 17.2 Control Applied to Analog model input and Model Response

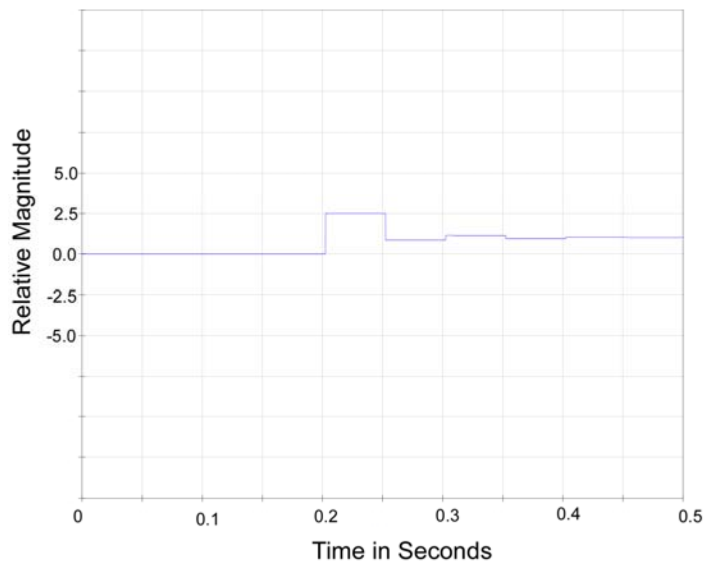

Figure 17.3 Control Signal Applied to Analog model input

18.0 Summary and Conclusions–

A controller was implemented using the deadbeat methodology. All equations were developed in the time domain using the SISO system model for the analog components. A third-order analog system was chosen to illustrate the high-order behaviors with one real pole and a complex-conjugate, underdamped pole pair with the understanding that higher-order systems can be identified as being composed of these components.

Both analog and digital system models were developed and the need for the Single-Input/Single-Output (SISO) form was developed. Good agreement was obtained from comparisons of each odel representation.

The time-shift "q" operator was introduced to develop a "rational" form for the SISO analog and digital models. The results were developed and compared to show good agreement between the models.

The time-shift "q" operator was used to develop a predictor for various "step-ahead" forms of the rational SISO model.

An error predictor was constructed to produce a prediction of tracking error.

Difference equations were written to represent first and second-order time derivatives of the predicted error behavior of the SISO digital model.

Simultaneous equations were solved to determine the sequence of control signals required to bring error and its time-derivatives to zero.

The sequence of control signals was applied to the digital model, and in parallel to the analog model to illustrate the effectiveness of the control methodology.

The controller showed the promised behavior exactly when controlling the digital model, but showed a damped oscillatory "ringing" behavior between sample periods for the analog system excited by the same control signal.

The controller showed very high control effort required to obtain faster response times.

Use of the control methodology for high sampling rates produced a controller that was ineffective for higher sampling rates, but very effective at slower sampling rates.

The methodology showed a simple deductive process for developing the controller from the pole locations of the analog model to be controlled.

The methodology showed that a relatively fast settling time for a step input could be achieved.

